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Preconditioners for ill-conditioned block Toeplitz systems with application in image restoration

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Abstract — The convergence rate of the conjugate gradient (CG) method for solving ill-conditioned Toeplitz systems is slow usually. To deal with such kind of systems, Potts and Steidl proposed in [11] an $\{\omega\}$-circulant preconditioner for solving these systems by using the preconditioned conjugate gradient (PCG) method. It was showed that the convergence rate of the PCG method is superlinear and the total operation cost is $O(N \log N)$, where N is the size of the system. Furthermore, it was suggested that the $\{\omega\}$-circulant preconditioner can also work efficiently for solving block Toeplitz systems. In this paper, we generalize this preconditioner for block systems. Both theoretical and numerical results are given. Besides, we apply this block preconditioner in image restoration.

Keywords: PCG method, block Toeplitz system, convergence rate, clustered spectrum, image restoration.

AMS(MOS) subject classification: 65F10, 65F15.

In this paper, we employ the PCG method to solve MN-by-MN linear system $A_{MN}x = b$ where A_{MN} is of the following form:

$$A_{MN} = \begin{bmatrix} A_0 & A_{-1} & \cdots & A_{2-M} & A_{1-M} \\ A_1 & A_0 & A_{-1} & \ddots & A_{2-M} \\ \vdots & A_1 & \ddots & \ddots & \vdots \\ A_{M-2} & \ddots & \ddots & A_0 & A_{-1} \\ A_{M-1} & A_{M-2} & \cdots & A_1 & A_0 \end{bmatrix}$$

with

$$A_j = \begin{bmatrix} a_0^{(j)} & a_{-1}^{(j)} & \cdots & a_{2-N}^{(j)} & a_{1-N}^{(j)} \\ a_1^{(j)} & a_0^{(j)} & a_{-1}^{(j)} & \ddots & a_{2-N}^{(j)} \\ \vdots & a_1^{(j)} & \ddots & \ddots & \vdots \\ a_{N-2}^{(j)} & \ddots & \ddots & a_0^{(j)} & a_{-1}^{(j)} \\ a_{N-1}^{(j)} & a_{N-2}^{(j)} & \cdots & a_1^{(j)} & a_0^{(j)} \end{bmatrix}, \quad j = 0, \pm 1, \ldots, \pm(M-1).$$

We call A_{MN} the block-Toeplitz-Toeplitz-block (BTTB) matrix. This kind of systems occurs in a variety of applications in mathematics and engineering such as the digital image processing and the discretization of two-dimensional partial differential equations, see [4]. We assume that the matrix A_{MN} is generated by a generating function $f = f(x, y)$, i.e., the diagonals of A_{MN} are given by the Fourier coefficients of f.

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If the generating function f is positive, the matrix A_{MN} generated by f is well-conditioned, see [2]. For the well-conditioned BTTB systems, the block-circulant-circulant-block (BCCB) matrices were proposed as preconditioners to solve the systems by the PCG method in [2] and [5]. It was showed that the eigenvalues of the preconditioned systems are clustered around 1 except $\mathcal{O}(M) + \mathcal{O}(N)$ outliers and the spectra are uniformly bounded in a positive interval. Hence the convergence rate of the PCG method is linear. If the generating function f has zeros, the matrix A_{MN} generated by f is ill-conditioned since the smallest eigenvalue tends to zero as the matrix size tends to infinity, see [12]. In this case, the BCCB preconditioners do not work efficiently, see the numerical results in [10]. To deal with this problem, Jin [7], Ng [10] and Serra [12] proposed using band block Toeplitz matrices with band Toeplitz blocks as preconditioners. It was showed that the condition numbers of the preconditioned systems are uniformly bounded and hence the PCG method converges linearly.

Recently, Potts and Steidl in [11] proposed an $\{\omega\}$-circulant preconditioner for solving ill-conditioned Toeplitz systems (in point case) by using the PCG method. It was proved that the PCG method converges superlinearly. Futhermore, it was suggested that this preconditioner is also good in solving the BTTB systems, see the numerical results in [11]. In this paper, we are going to generalize the $\{\omega\}$-circulant preconditioner for block systems. Both theoretical result on the convergence rate of the PCG method and numerical results are given. Besides, we apply this block preconditioner in image restoration.

The outline of this paper is as follows. In Section 1, we generalize the preconditioners proposed in [11] for block systems. In Section 2, we show that the eigenvalues of the preconditioned matrices are clustered around 1 except $\mathcal{O}(M) + \mathcal{O}(N)$ outliers. In Section 3, we give the convergence rate and operation cost of the PCG method with our block preconditioners. Finally, numerical results and an application in image restoration are given in Sections 4 and 5.

1. CONSTRUCTION OF PRECONDITIONERS

Let $C_{2\pi\times 2\pi}$ be the space of all 2π-periodic continuous real-valued functions with two variables and $A_{MN}(f)$ be the MN-by-MN BTTB matrix generated by the function $f \in C_{2\pi\times 2\pi}$. More precisely, the entries of $A_{MN}(f)$ are given by the Fourier coefficients of f, i.e.,

$$a_{j,k}(f) = a_k^{(j)}(f) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)e^{-i(jx+ky)}\,dx\,dy, \qquad j,k = 0,\pm 1,\pm 2,\ldots.$$

Since f is real-valued, the matrix $A_{MN}(f)$ is Hermitian, see Lemma 2.1 in [12]. We study the solution of the BTTB system

$$A_{MN}(f)x = b \tag{1.1}$$

where $A_{MN}(f)$ is generated by a nonnegative function f having finite number of zeros.

Let $u \in \mathbb{C}^{MN}$ with $u^H u = 1$ and

$$u = (u_0^t, u_1^t, u_2^t, \ldots, u_{M-1}^t)^t, \qquad u_j \in \mathbb{C}^N, \qquad j = 0,1,\ldots,M-1\,.$$

Since

$$u^H A_{MN}(f)u = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\left|\sum_{j=0}^{M-1}\sum_{k=0}^{N-1}(u_j)_k\, e^{-i(jx+ky)}\right|^2 dx\,dy \tag{1.2}$$

where $(u_j)_k$ denotes the k-th component of the vector u_j, the BTTB matrix $A_{MN}(f)$ is positive semidefinite, see [6]. Moreover, we have

Lemma 1.1 (Serra [12]). *Let $f \in C_{2\pi\times 2\pi}$ be nonnegative. If f is not identically zero, then the matrix $A_{MN}(f)$ is positive definite.*

Since $A_{MN}(f)$ is positive definite, the system (1.1) could be solved by applying the PCG method. It is well-known that the convergence rate of the PCG method depends on the spectrum of the preconditioned matrix. The more clustered the spectrum, the faster the convergence rate will be. Now, we are going to construct a positive definite preconditioner $\mathcal{T}_{MN}(f)$ of $A_{MN}(f)$ such that the preconditioned matrix $\mathcal{T}_{MN}(f)^{-1}A_{MN}(f)$ has clustered spectrum.

Let the grid points (x_r, y_s) be given by

$$x_r = \frac{2\pi r}{M} + \xi - \pi \quad \text{and} \quad y_s = \frac{2\pi s}{N} + \eta - \pi$$

where $(\xi, \eta) \in [0, \frac{2\pi}{M}) \times [0, \frac{2\pi}{N})$, $r = 0, 1, \ldots, M-1$ and $s = 0, 1, \ldots, N-1$. Since f has only a finite number of zeros, we can choose a suitable pair $(\xi, \eta) \in [0, \frac{2\pi}{M}) \times [0, \frac{2\pi}{N})$ such that

$$f(x_r, y_s) > 0\,. \tag{1.3}$$

To approximate the integral on the right-hand side of (1.2) by using the trapezoidal rule with respect to the above grid points, we obtain

$$\begin{aligned}
u^H A_{MN}(f)u &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\left|\sum_{j=0}^{M-1}\sum_{k=0}^{N-1}(u_j)_k\, e^{-\mathrm{i}(jx+ky)}\right|^2 \mathrm{d}x\,\mathrm{d}y \\
&\approx \frac{1}{MN}\sum_{r=0}^{M-1}\sum_{s=0}^{N-1}\left|\sum_{j=0}^{M-1}\sum_{k=0}^{N-1}(u_j)_k\, e^{-\mathrm{i}(jx_r+ky_s)}\right|^2 f(x_r, y_s) \qquad (1.4)\\
&= \sum_{r=0}^{M-1}\sum_{s=0}^{N-1} f(x_r, y_s)\frac{1}{\sqrt{MN}}\left(\sum_{j=0}^{M-1}\sum_{k=0}^{N-1}(\bar{u}_j)_k\, e^{-2\pi\mathrm{i}(jr/M+ks/N)}\, e^{-\mathrm{i}[j(\xi-\pi)+k(\eta-\pi)]}\right)\\
&\quad\times\frac{1}{\sqrt{MN}}\left(\sum_{j=0}^{M-1}\sum_{k=0}^{N-1}(u_j)_k\, e^{2\pi\mathrm{i}(jr/M+ks/N)}\, e^{\mathrm{i}[j(\xi-\pi)+k(\eta-\pi)]}\right)\\
&= (\mathcal{F}_{MN}\mathcal{W}_{MN}\bar{u})'\mathcal{D}_{MN}\bar{\mathcal{F}}_{MN}\bar{\mathcal{W}}_{MN}u = u^H\mathcal{T}_{MN}(f)u\,.
\end{aligned}$$

Here $\mathcal{W}_{MN}$ and $\mathcal{D}_{MN}$ are diagonal matrices defined as follows,

$$\mathcal{W}_{MN} = W_M^{\xi} \otimes W_N^{\eta}$$

$$W_M^{\xi} = \operatorname{diag}\left((e^{-\mathrm{i}j(\xi-\pi)})_{j=0}^{M-1}\right), \quad W_N^{\eta} = \operatorname{diag}\left((e^{-\mathrm{i}j(\eta-\pi)})_{j=0}^{N-1}\right)$$

$$\mathcal{D}_{MN} = \operatorname{diag}(D_0, D_1, \ldots, D_{M-1})$$

$$D_i = \operatorname{diag}\left((f(x_i, y_j))_{j=0}^{N-1}\right), \quad i = 0, 1, \ldots, M-1$$

the $\mathcal{F}_{MN}$ is the two-dimensional Fourier matrix given by

$$\mathcal{F}_{MN} = F_M \otimes F_N, \quad F_J = \frac{1}{\sqrt{J}}\left(e^{-2\pi\mathrm{i}jk/J}\right)_{j,k=0}^{J-1}, \quad J = M, N$$

and

$$\mathcal{T}_{MN}(f) = \mathcal{W}_{MN}\mathcal{F}_{MN}\mathcal{D}_{MN}\bar{\mathcal{F}}_{MN}\bar{\mathcal{W}}_{MN}\,. \tag{1.5}$$

By using (1.3) and (1.5), the matrix $\mathcal{T}_{MN}(f)$ is Hermitian and positive definite. Let $v = \mathcal{T}_{MN}(f)^{1/2}u$, we get

$$v^H \mathcal{T}_{MN}(f)^{-1/2} A_{MN}(f) \mathcal{T}_{MN}(f)^{-1/2} v \approx v^H v\,.$$

By using the two-dimensional fast Fourier transform (FFT), any matrix-vector multiplication

$$\mathcal{T}_{MN}(f)^{-1} v = \mathcal{W}_{MN} \mathcal{F}_{MN} \mathcal{D}_{MN}^{-1} \bar{\mathcal{F}}_{MN} \bar{\mathcal{W}}_{MN} v$$

requires at most $\mathcal{O}(MN \log MN)$ arithmetical operations. Therefore $\mathcal{T}_{MN}(f)$ seems to be a good preconditioner of $A_{MN}(f)$. In the next section, we will prove that the eigenvalues of $\mathcal{T}_{MN}(f)^{-1} A_{MN}(f)$ are clustered around 1 except $\mathcal{O}(M) + \mathcal{O}(N)$ outliers.

2. CLUSTERING OF EIGENVALUES

In Section 1, we prove that $u^H A_{MN}(f) u \approx u^H \mathcal{T}_{MN}(f) u$. Now, we consider the approximation error. Rewrite (1.4) as

$$u^H A_{MN}(f) u = \sum_{j,k=0}^{M-1} \sum_{p,q=0}^{N-1} (\bar{u}_j)_p a_{j-k,p-q} (u_k)_q \approx \sum_{j,k=0}^{M-1} \sum_{p,q=0}^{N-1} (\bar{u}_j)_p \bar{a}_{j-k,p-q} (u_k)_q = u^H \mathcal{T}_{MN}(f) u \tag{2.1}$$

where

$$\bar{a}_{j,k} = \bar{a}_{j,k}(f) = \frac{1}{MN} \sum_{r=0}^{M-1} \sum_{s=0}^{N-1} f(x_r, y_s) e^{-2\pi \mathrm{i}(jr/M + ks/N)}\, e^{-\mathrm{i}[j(\xi-\pi)+k(\eta-\pi)]} \tag{2.2}$$

for $j, k = 0, \pm 1, \pm 2, \ldots$. The approximation error depends on $a_{j,k}$ and $\bar{a}_{j,k}$. Now replacing $f(x_r, y_s)$ by the Fourier series of f at (x_r, y_s) in (2.2), we obtain

$$\begin{aligned} \bar{a}_{j,k} &= \frac{1}{MN} \sum_{r=0}^{M-1} \sum_{s=0}^{N-1} \left(\sum_{p,q \in \mathbb{Z}} a_{p,q}\, e^{\mathrm{i}(px_r + qy_s)} \right) e^{-2\pi \mathrm{i}(jr/M+ks/N)}\, e^{-\mathrm{i}[j(\xi-\pi)+k(\eta-\pi)]} \\ &= a_{j,k} + \sum_{\alpha,\beta \in \mathbb{Z}\setminus\{0\}} a_{j+\alpha M, k+\beta N}\, e^{\mathrm{i}[\alpha M(\xi-\pi)+\beta N(\eta-\pi)]} \\ &\quad + \sum_{\alpha \in \mathbb{Z}\setminus\{0\}} a_{j+\alpha M,k}\, e^{\mathrm{i}\alpha M(\xi-\pi)} + \sum_{\beta \in \mathbb{Z}\setminus\{0\}} a_{j,k+\beta N}\, e^{\mathrm{i}\beta N(\eta-\pi)}\,. \end{aligned} \tag{2.3}$$

Let

$$\begin{aligned} b_{j,k} = b_{j,k}(f) &= \sum_{\alpha,\beta \in \mathbb{Z}\setminus\{0\}} a_{j+\alpha M, k+\beta N}\, e^{\mathrm{i}[\alpha M(\xi-\pi)+\beta N(\eta-\pi)]} \\ &\quad + \sum_{\alpha \in \mathbb{Z}\setminus\{0\}} a_{j+\alpha M,k}\, e^{\mathrm{i}\alpha M(\xi-\pi)} + \sum_{\beta \in \mathbb{Z}\setminus\{0\}} a_{j,k+\beta N}\, e^{\mathrm{i}\beta N(\eta-\pi)} \end{aligned} \tag{2.4}$$

then it follows by using (2.3) and the definition of $A_{MN}(f)$ that

$$A_{MN}(f) = \mathcal{T}_{MN}(f) - B_{MN}(f)\,. \tag{2.5}$$

Here $B_{MN}(f)$ is the BTTB matrix determined by $b_{j,k}$, i.e., the (r,s)-th entry of the (p,q)-th block of $B_{MN}(f)$ is given by $b_{p-q,r-s}$. Thus

$$\mathcal{T}_{MN}(f)^{-1} A_{MN}(f) = I_{MN} - \mathcal{T}_{MN}(f)^{-1} B_{MN}(f) \tag{2.6}$$

where I_{MN} is the MN-by-MN identity matrix.

Lemma 2.1. *Let $p_{s,t}$ be a nonnegative trigonometric polynomial of the form*

$$\sum_{p=-s}^{s} \sum_{q=-t}^{t} \alpha_{p,q} \, e^{\mathrm{i}(px+qy)}$$

where $\alpha_{p,q}$ are constants. If $2s \leq M$ and $2t \leq N$, then there are at most $\mathcal{O}(M) + \mathcal{O}(N)$ eigenvalues of $\mathcal{T}_{MN}(p_{s,t})^{-1} A_{MN}(p_{s,t})$ different from 1.

Proof. By using the definition of $a_{j,k}$, it follows that $a_{j+\alpha M,k+\beta N}(p_{s,t}) = 0$ for $|j| \leq M-1-s$ or $|k| \leq N-1-t$, $a_{j+\alpha M,k}(p_{s,t}) = 0$ for $|j| \leq M-1-s$ or $|k| \geq t+1$, and $a_{j,k+\beta N}(p_{s,t}) = 0$ for $|j| \geq s+1$ or $|k| \leq N-1-t$. By using (2.4), the rank of the matrix $B_{MN}(p_{s,t})$ is $\mathcal{O}(M) + \mathcal{O}(N)$. Now, the result follows from (2.6). □

For the proof of our main theorem, we need the following lemma.

Lemma 2.2. *Let $g \in \mathcal{C}_{2\pi\times 2\pi}$ be a nonnegative function with finite number of zeros and $h \in \mathcal{C}_{2\pi\times 2\pi}$ be a positive function with $h_{\min} > 0$. Define $f = gh$. Then, for any $M, N \in \mathbb{N}$, the eigenvalues of $A_{MN}(g)^{-1} A_{MN}(f)$ lie in the interval $[h_{\min}, h_{\max}]$.*

Proof. By the mean value theorem of integration, we have

$$\begin{aligned} u^H A_{MN}(f) u &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y) \left| \sum_{j=0}^{M-1}\sum_{k=0}^{N-1} (u_j)_k \, e^{-\mathrm{i}(jx+ky)} \right|^2 dx\,dy \\ &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(x,y)h(x,y) \left| \sum_{j=0}^{M-1}\sum_{k=0}^{N-1} (u_j)_k \, e^{-\mathrm{i}(jx+ky)} \right|^2 dx\,dy \\ &= h_* \frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(x,y) \left| \sum_{j=0}^{M-1}\sum_{k=0}^{N-1} (u_j)_k \, e^{-\mathrm{i}(jx+ky)} \right|^2 dx\,dy = h_* u^H A_{MN}(g) u \end{aligned}$$

where $h_* \in [h_{\min}, h_{\max}]$. Since the matrix $A_{MN}(g)$ is positive definite, we have

$$h_{\min} \leq h_* = \frac{u^H A_{MN}(f) u}{u^H A_{MN}(g) u} \leq h_{\max}$$

for any nonzero vector $u \in \mathbb{C}^{MN}$. By the properties of Rayleigh quotient, the eigenvalues of $A_{MN}(g)^{-1} A_{MN}(f)$ lie in the interval $[h_{\min}, h_{\max}]$. □

Remark 2.1. The above lemma was proved first by Serra in [12]. Here we give another simpler proof.

Definition 2.1 (Ng [10]). Suppose that (x_0, y_0) is a zero of $f(x,y)$. We define the following function $F_{(x_0,y_0)}(\tau)$ by

$$F_{(x_0,y_0)}(\tau) = f(x_0 + \tau(x - x_0), y_0 + \tau(y - y_0)) \quad \forall\,(x,y) \neq (x_0, y_0)\,.$$

Then the zero (x_0, y_0) is of order n if n is the smallest positive integer such that the $(n+1)$-th derivative $F^{(n+1)}_{(x_0,y_0)}(\tau)$ is continuousin a neighborhood of (x_0, y_0) and $F^{(n)}_{(x_0,y_0)}(\tau) \neq 0$ for all $(x,y) \neq (x_0, y_0)$.

It was proved in [10] that if f is a nonnegative function with zero (x_0, y_0) of order n, $F^{(n)}_{(x_0,y_0)}(0)$ is positive for all $(x, y) \neq (x_0, y_0)$ and n must be even. In the following, we restrict our attention to nonnegative generating functions $f \in \mathcal{C}_{2\pi\times 2\pi}$ with zeros attained at

$$t_1, t_2, \ldots, t_k \in (-\pi, \pi) \times (-\pi, \pi).$$

Here $t_i = (x_i, y_i)$ and the order of each zero t_i is $2\nu_i$, $i = 1, 2, \ldots, k$.

Let

$$p(t) = p(x, y) = \prod_{i=1}^{k} (2 - \cos(x - x_i) - \cos(y - y_i))^{\nu_i} \tag{2.7}$$

then p has zeros attained at t_i of even order for $i = 1, 2, \ldots, k$. Moreover, this trigonometric polynomial matches the zero of f, i.e., f/p is continuous and greater than zero on $[-\pi, \pi) \times [-\pi, \pi)$, see [10]. Now, we are ready to prove the main theorem.

Theorem 2.1. *Let $f \in \mathcal{C}_{2\pi\times 2\pi}$ be a nonnegative function having finite number of zeros of even order. Then the eigenvalues of $\mathcal{T}_{MN}(f)^{-1}A_{MN}(f)$ are clustered around 1 except $\mathcal{O}(M) + \mathcal{O}(N)$ outliers.*

Proof. Let $h = f/p$. Then h is continuous and positive on $[-\pi, \pi) \times [-\pi, \pi)$. Moreover, since the set of trigonometric polynomials is dense in $\mathcal{C}_{2\pi\times 2\pi}$, for any $\epsilon > 0$, there exists a positive trigonometric polynomial $g(x, y)$ such that

$$g(x, y) - \frac{1}{2}\epsilon\, h_{\min} \le h(x, y) \le g(x, y) + \frac{1}{2}\epsilon\, h_{\min} \tag{2.8}$$

for all $(x, y) \in [-\pi, \pi) \times [-\pi, \pi)$. Since $p \ge 0$, we have

$$gp - \frac{1}{2}\epsilon\, h_{\min}p \le f \le gp + \frac{1}{2}\epsilon\, h_{\min}p. \tag{2.9}$$

For the right-hand side in the above inequality, we obtain by using (1.2),

$$u^H A_{MN}(f)u \le u^H A_{MN}(gp)u + \frac{1}{2}\epsilon h_{\min} u^H A_{MN}(p)u.$$

Further, since $\mathcal{T}_{MN}(f)$ is positive definite, we have

$$\frac{u^H A_{MN}(f)u}{u^H \mathcal{T}_{MN}(f)u} \le \frac{u^H A_{MN}(gp)u}{u^H \mathcal{T}_{MN}(f)u} + \frac{1}{2}\epsilon\, h_{\min}\frac{u^H A_{MN}(p)u}{u^H \mathcal{T}_{MN}(f)u} \tag{2.10}$$

for all nonzero $u \in \mathbb{C}^{MN}$. Now, it holds by (2.5) and Lemma 2.1 that

$$A_{MN}(p) = \mathcal{T}_{MN}(p) + \tilde{R}_{MN} \tag{2.11}$$

and

$$A_{MN}(gp) = \mathcal{T}_{MN}(gp) + \hat{R}_{MN} \tag{2.12}$$

where

$$\operatorname{rank}(\tilde{R}_{MN}) = \operatorname{rank}(\hat{R}_{MN}) = \mathcal{O}(M) + \mathcal{O}(N).$$

Substituting (2.11) and (2.12) into (2.10), we obtain

$$\frac{u^H A_{MN}(f)u}{u^H \mathcal{T}_{MN}(f)u} \le \frac{u^H \mathcal{T}_{MN}(gp)u}{u^H \mathcal{T}_{MN}(f)u} + \frac{u^H \hat{R}_{MN}u}{u^H \mathcal{T}_{MN}(f)u} + \frac{1}{2}\epsilon\, h_{\min}\frac{u^H \mathcal{T}_{MN}(p)u}{u^H \mathcal{T}_{MN}(f)u} + \frac{1}{2}\epsilon\, h_{\min}\frac{u^H \tilde{R}_{MN}u}{u^H \mathcal{T}_{MN}(f)u}.$$

Since

$$\frac{u^H \mathcal{T}_{MN}(p)u}{u^H \mathcal{T}_{MN}(f)u} = \frac{1}{u^H \mathcal{T}_{MN}(h)u} \leq \frac{1}{h_{\min}}$$

we get

$$\frac{u^H (A_{MN}(f) - R_{MN})u}{u^H \mathcal{T}_{MN}(f)u} \leq \frac{u^H \mathcal{T}_{MN}(gp)u}{u^H \mathcal{T}_{MN}(f)u} + \frac{1}{2}\epsilon$$

where

$$R_{MN} = \frac{1}{2}\epsilon\, h_{\min}\bar{R}_{MN} + \hat{R}_{MN}$$

with $\text{rank}(R_{MN}) = \mathcal{O}(M) + \mathcal{O}(N)$. Now, by using the facts

$$\mathcal{T}_{MN}(gp) = \mathcal{T}_{MN}(g)\mathcal{T}_{MN}(p) \quad \text{and} \quad \mathcal{T}_{MN}(f) = \mathcal{T}_{MN}(h)\mathcal{T}_{MN}(p)$$

we have

$$\frac{u^H (A_{MN}(f) - R_{MN})u}{u^H \mathcal{T}_{MN}(f)u} \leq \frac{u^H \mathcal{T}_{MN}(g)u}{u^H \mathcal{T}_{MN}(h)u} + \frac{1}{2}\epsilon . \tag{2.13}$$

Let $v = \mathcal{T}_{MN}(p)^{1/2}u$ for all nonzero $u \in \mathbb{C}^{MN}$. By using (2.8), we have

$$v^H \mathcal{T}_{MN}(g)v \leq v^H \mathcal{T}_{MN}(h)v + \frac{1}{2}\epsilon\, h_{\min} v^H v .$$

Since, by using Lemma 2.2,

$$0 < \frac{v^H v}{v^H \mathcal{T}_{MN}(h)v} = \frac{u^H \mathcal{T}_{MN}(p)u}{u^H \mathcal{T}_{MN}(f)u} \leq \frac{1}{h_{\min}}$$

we further have

$$\frac{u^H \mathcal{T}_{MN}(g)u}{u^H \mathcal{T}_{MN}(h)u} \leq 1 + \frac{1}{2}\epsilon .$$

Using the above inequality in (2.13), we obtain

$$\frac{u^H (A_{MN}(f) - R_{MN})u}{u^H \mathcal{T}_{MN}(f)u} \leq 1 + \epsilon .$$

Similarly, we conclude from the left-hand inequality of (2.9) that

$$\frac{u^H (A_{MN}(f) - R_{MN})u}{u^H \mathcal{T}_{MN}(f)u} \geq 1 - \epsilon .$$

Consequently, since

$$\text{rank}(R_{MN}) = \mathcal{O}(M) + \mathcal{O}(N)$$

we know that at most $\mathcal{O}(M) + \mathcal{O}(N)$ eigenvalues of $\mathcal{T}_{MN}(f)^{-1}A_{MN}(f)$ are not contained in $[1-\epsilon, 1+\epsilon]$. This completes the proof. □

Hence, if the PCG method is applied to solve the BTTB system (1.1), we can expect a fast convergence rate.

3. CONVERGENCE RATE AND OPERATION COST

In this section, following the idea in [8], we analyze the convergence rate and operation cost of solving the BTTB systems by using the PCG method with preconditioner $\mathcal{T}_{MN}$. For simplicity, let $M = N$. We first show that the method converges within $\mathcal{O}(\alpha N \log N)$ steps if $\lambda_{\min}(\mathcal{T}_{NN}(f)^{-1}A_{NN}(f)) = \mathcal{O}(N^{-\alpha})$ where $\lambda_{\min}(\mathcal{T}_{NN}(f)^{-1}A_{NN}(f))$ is the smallest eigenvalue of $\mathcal{T}_{NN}(f)^{-1}A_{NN}(f)$ and $\alpha > 0$. We begin by introducing the following lemma, see [1].

Lemma 3.1. *Let $G \in \mathbb{C}^{N\times N}$ be a positive definite matrix and x be the solution of the system $Gx = b$. Let x_j be the j-th iterant of the CG method applied to $Gx = b$. If the eigenvalues $\{\lambda_k\}$ of G are distributed such that*

$$0 < \lambda_1 \leq \ldots \leq \lambda_p \leq d_1 \leq \lambda_{p+1} \leq \ldots \leq \lambda_{N-q} \leq d_2 \leq \lambda_{N-q+1} \leq \ldots \leq \lambda_N$$

then

$$\frac{\|x_j - x\|_G}{\|x_0 - x\|_G} \leq 2\left(\frac{d-1}{d+1}\right)^{j-p-q} \cdot \max_{\lambda\in[d_1,d_2]}\left\{\prod_{k=1}^{p}\left(\frac{\lambda-\lambda_k}{\lambda_k}\right)\right\}. \tag{3.1}$$

Here $d = (d_2/d_1)^{1/2} \geq 1$ and $\|v\|_G = v^H G v$.

For solving the preconditioned system of (1.1), we take $G = \mathcal{T}_{NN}(f)^{-1}A_{NN}(f)$. By Theorem 2.1, we can choose $d_1 = 1-\epsilon$ and $d_2 = 1+\epsilon$. Then $p+q = \mathcal{O}(N)$. By choosing $\epsilon < 1$, we have

$$(d-1)/(d+1) = (1-\sqrt{1-\epsilon^2})/\epsilon < \epsilon\,. \tag{3.2}$$

In order to use (3.1), we note that

$$\|G^{-1}\|_2 = \lambda_{\min}^{-1}(G) = \lambda_{\min}^{-1}(\mathcal{T}_{NN}(f)^{-1}A_{NN}(f)) \leq c_1 N^{\alpha}$$

for some $c_1 > 0$ independent of N. Therefore for each k,

$$\lambda_k \geq \min_l \lambda_l = \|G^{-1}\|_2^{-1} \geq c_2 N^{-\alpha}$$

where $c_2 = c_1^{-1}$. Thus for $1 \leq k \leq p$ and $\lambda \in [1-\epsilon, 1+\epsilon]$, we have

$$0 \leq \frac{(\lambda-\lambda_k)}{\lambda_k} \leq c_3 N^{\alpha}$$

for some $c_3 > 0$ independent of N. Combining this with (3.2), then (3.1) becomes

$$\|x - x_j\|_G/\|x - x_0\|_G \leq c N^{p\alpha}\epsilon^{j-p-q}$$

where $c > 0$ is a constant independent of N. Given an arbitrary tolerance $\tau > 0$, an upper bound for the number of iterations required to make

$$\|x - x_j\|_G/\|x - x_0\|_G \leq \tau$$

is given by

$$j_0 = p + q - (p\log c + \alpha p \log N - \log\tau)/\log\epsilon \leq \mathcal{O}(\alpha N \log N)\,.$$

Now, let us consider the operation cost of the PCG method in each iteration. The main work involved in implementing the PCG method is the matrix-vector multiplication $\mathcal{T}_{NN}(f)^{-1}A_{NN}(f)v$ for some vector v. For the product $A_{NN}(f)v$, we embed the matrix $A_{NN}(f)$ into a $4N^2$-by-$4N^2$ BCCB matrix. Then $A_{NN}(f)v$ can be carried out within $\mathcal{O}(N^2 \log N)$ operations by using two-dimensional FFT, see [2]. For the multiplication $\mathcal{T}_{NN}(f)^{-1}y$ where y is a vector, according to the last part in Section 1, it can be computed not more than $\mathcal{O}(N^2 \log N)$ operations. Thus, the cost per iteration of the PCG method is $\mathcal{O}(N^2 \log N)$. Combining the above results, the total complexity of our method is $\mathcal{O}(\alpha N^3 \log^2 N)$ if $\lambda_{\min}(G) = \mathcal{O}(N^{-\alpha})$.

4. NUMERICAL RESULTS

We apply the PCG method with our preconditioner for solving the BTTB system

$$A_{NN}(f)x = b$$

where $A_{NN}(f)$ is generated by a nonnegative function f having finite number of zeros. We test the following three systems with different generating functions defined on $[-\pi, \pi) \times [-\pi, \pi)$. They are

$$\begin{array}{ll} \text{(i)} & f_1(x,y) = x^2 + y^2 \\ \text{(ii)} & f_2(x,y) = x^2 + y^4 \\ \text{(iii)} & f_3(x,y) = (x^2-1)^2 y^2 . \end{array}$$

The systems (i) and (ii) have a zero at (0,0) and system (iii) has zeros at (1,0) and (-1,0). In all tests, we used the vector of all ones as the right hand side vector b and the zero vector as the initial guess of the solution x. The stopping criteria is $\|r_k\|_2/\|r_0\|_2 < 10^{-7}$ where r_k is the residual vector after k-th iterations. All computations are produced by Matlab program.

Tables 1 to 3 show the numbers of iterations required for convergence with different choices of preconditioners. In these tables, I denotes no preconditioner and $\mathcal{T}_{NN}(f_i)$, $i = 1, 2, 3$, denotes our preconditioner. Iteration numbers greater than 10000 are denoted by "-". For comparisons, the BCCB preconditioner proposed in [2] and [5] is also tested. From the numerical results, there is evident that the convergence performance of our preconditioner is better than the BCCB preconditioner.

We would like to remark that if $f(x,y) = g(x)h(y)$, as given in system (iii), then the corresponding matrix is a block Toeplitz matrix with tensor structure. We could use the fast algorithm proposed in [9] to obtain a faster convergence rate.

Table 1.
Number of iterations for $f_1(x,y) = x^2 + y^2$.

N	N^2	I	BCCB	$\mathcal{T}_{NN}(f_1)$
8	64	10	10	7
16	256	32	14	11
32	1024	75	20	11
64	4096	161	29	13
128	16384	333	46	16
256	65536	681	73	16

Table 2.
Number of iterations for $f_2(x,y) = x^2 + y^4$.

N	N^2	I	BCCB	$\mathcal{T}_{NN}(f_2)$
8	64	19	14	12
16	256	95	28	16
32	1024	291	56	26
64	4096	781	122	37
128	16384	2032	267	60
256	65536	4958	621	101

Table 3.
Number of iterations for $f_3(x,y) = (x^2-1)^2y^2$.

N	N^2	I	BCCB	$\mathcal{T}_{NN}(f_3)$
8	64	37	18	21
16	256	359	64	50
32	1024	2608	125	34
64	4096	–	271	45
128	16384	–	559	73
256	65536	–	1260	71

5. IMAGE RESTORATION

The mathematical model of the linear image restoration problem is given as follows,

$$g(\xi,\delta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t(\xi-\alpha,\delta-\beta)f(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta + \eta(\xi,\delta) \tag{5.1}$$

where $g(\xi,\delta)$ is the degraded image, $f(\alpha,\beta)$ is the original image and the vector $\eta(\xi,\delta)$ is the additive noise. The function $t(\xi-\alpha,\delta-\beta)$ is the point spread function (PSF) and represents the degradation of the image.

In the digital implementation of (5.1), the integral is discretized to obtain the discrete scalar model

$$g(i,j) = \sum_{k=1}^{N}\sum_{l=1}^{N} t(i-k,j-l)f(k,l) + \eta(i,j)\,.$$

In matrix form, we have

$$g = Tf + \eta$$

where g, η, f are vectors of size N^2 and T is an N^2-by-N^2 BTTB matrix, see [3].

The following example is constructed. We generate the known 128×128 image of Lena, see Fig. 1 (left), and consider the discretized PSF matrix T with the entries given by

$$t_{i-j,k-l} = \begin{cases} \exp\{-0.5(i-j)^2 - 0.5(k-l)^2\}, & -8 \le i-j, k-l \le 8 \\ 0 & \text{otherwise.} \end{cases}$$

We construct the observed image, see Fig. 1 (right), by forming the vector $g = Tf + \eta$ and where the entries of η is chosen from a standard normal distribution and is scaled such that $\|\eta\|_2/\|Tf\|_2 = 10^{-3}$.

Our objective is to recover an approximation to the original image f when g and T are given. We give the number of iterations required for convergence of our preconditioner $\mathcal{T}_{NN}$. The stopping criteria is $\|r_k\|_2/\|r_0\|_2 < 10^{-5}$.

Figure 1. Original image: Lena (left) and observed image (right).

Figure 2. Restored image with $\mathcal{T}_{NN}$: 2 (left) and 7 iterations (right).

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