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On convergence properties of 3D spheroidal monogenics

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J. Morais has recently introduced certain complete orthogonal sets of monogenic polynomials over 3D prolate spheroids with remarkable properties. The underlying functions take on either values in the reduced and full quaternions (identified, respectively, with $\mathbb{R}^3$ and $\mathbb{R}^4$), and are generally assumed to be nullsolutions of the well known Riesz and Moisil-Théodoresco systems in $\mathbb{R}^3$. In continuation of these studies, we recall some fundamental properties of the polynomials, and prove some recursive formulae between them. As a consequence, we obtain a two-term type recurrence relation satisfied by those basis polynomials. These results are then employed to investigate a rather wide class of approximation properties for monogenic functions over 3D prolate spheroids in terms of spheroidal monogenics.

Keywords: Quaternion analysis; Riesz system; Moisil-Théodoresco system; Ferrer's associated Legendre functions; Chebyshev polynomials; hyperbolic functions; monogenic functions.

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1. Introduction

Quaternion analysis is thought to generalize onto the multidimensional situation the classical theory of holomorphic functions of one complex variable, and to provide the foundations for a refinement of classical harmonic analysis. In the meantime, quaternion analysis has become a well established mathematical discipline and an active area of research with numerous connections to other areas of both pure and applied mathematics. In particular, there is a well developed theory of quaternion analysis with many applications to boundary value problems and partial differential equations theory, as well as to other fields of physics and engineering. A thorough treatment is listed in the bibliography, *e.g.* K. Gürlebeck and W. Sprößig [19,20], V. Kravchenko and M. Shapiro [27], V. Kravchenko [28], M. Shapiro and N. Vasilevski [46,47], and A. Sudbery [52]. The rich structure of this function theory involves the analysis of monogenic functions defined in open subsets of $\mathbb{R}^3$, which are nullsolutions of higher-dimensional Cauchy-Riemann systems. In this paper we investigate certain approximation properties for monogenic functions over 3D prolate spheroids. Our starting point is a spheroidal monogenic expansion in terms of monogenic polynomials which offer a refinement of the notion of spheroidal harmonics. The underlying spheroidal functions (C. Flammer [14], E. Hobson [25], N. Lebedev [30]) are interesting in their own right, forming a natural bridge between properties of the Legendre and Chebyshev polynomials.

Since the foundations of the theory of approximation of monogenic functions by R. Fueter [15,16], the study of orthogonal polynomials in application to certain boundary value problems for elliptic partial differential equations (such as the Maxwell, Lamé and Stokes system) has been of great importance in connection with certain problems of mathematical physics [3]. In our view much of the older theory has progressed considerably upon the study of monogenic polynomial approximations in the context of quaternion analysis (see *e.g.* [5,6,7,10,29,31,35,36,37,38]). For a detailed historic survey and extended list of references on monogenic approximations we refer the reader to [22]. The standard domain, in which the previous contributions are developed, is the ball. The progress of a function theory on a more general type of domains is still missing a part on the research front. Most relevant to our study are the intimate connections between monogenic functions and spheroidal structures, and the potential flexibility afforded by a spheroid's nonspherical canonical geometry. Developments are described in the sequence of papers by H. Malonek et al. in [32,33] (cf. [1,9]) and J. Morais et al. in [23,24,41]. In light of this, in [39,40] (cf. [17,42]) a very recent approach has been developed to discuss approximation properties for monogenic functions over 3D prolate spheroids by Fourier expansions in monogenic polynomials of which could be explicitly expressed in terms of products of Ferrer's associated Legendre functions multiplied by Chebyshev polynomial factors (see Theorem 3.1 below). Within the scope of this paper we shall be fully concerned with the polynomials introduced in these notes. Studies have shown that the underlying spheroidal monogenics play an important role in defining the monogenic Szegö kernel function for 3D prolate spheroids [43].

The outline of the paper is as follows. Section 2 gives a brief introduction to some general definitions of quaternion analysis. Section 3 reviews two distinct complete orthogonal sets of 3D prolate spheroidal monogenics. Some important properties, and efficient recurrence formulae for the basis polynomials are discussed. The gained insight is applied to certain approximation properties for monogenic functions in terms of those polynomials (Section 4). This will be done in the spaces of square integrable functions over $\mathbb{H}$. The main motivation of the present study is to develop further general numerical methods to solve both basic boundary value and conformal mapping problems. In a forthcoming article we shall describe these connections in more detail and illustrate them by some examples.

Interestingly, the used methods have an nD counterpart and can be either extended to oblate spheroids or even to arbitrary ellipsoids. But such a procedure would make the computations somewhat laborious, and for this reason we shall not discuss these general cases here. All the results obtained in this paper are new.

2. Preliminaries

2.1. *Prolate spheroidal harmonics*

In prolate spheroidal coordinates (see e.g. E. Hobson [25], N. Lebedev [30]), the Cartesian coordinates may be parameterized by $x = x(\mu, \theta, \phi)$, $\mu \in [0, \infty)$, $\theta \in [0, \pi)$, and $\phi \in [0, 2\pi)$, such that

$$x_0 = ca \cos\theta, \quad x_1 = cb \sin\theta \cos\phi, \quad x_2 = cb \sin\theta \sin\phi \tag{2.1}$$

where c is the prolatness parameter, and $a = \cosh\mu$, $b = \sinh\mu$, are respectively, the semimajor and semiminor axis of the generating ellipse. Using transformation relations (2.1) the surfaces of revolution for which μ is the parameter consist of the confocal prolate spheroids

$$\mathcal{S} : \frac{x_0^2}{c^2 \cosh^2\mu} + \frac{x_1^2 + x_2^2}{c^2 \sinh^2\mu} = 1. \tag{2.2}$$

Accordingly, the surface of $\mathcal{S}$ is matched with the surface of the supporting spheroid $\mu = \alpha$ if we put $c^2 \cosh^2\alpha = a^2$, and $c^2 \sinh^2\alpha = b^2$. Then we obtain the prolatness parameter $c = \sqrt{a^2 - b^2} \in (0, 1)$, which means that c is the eccentricity of the ellipse with foci on the x_0-axis: $(-c, 0, 0)$, $(+c, 0, 0)$.

The particular solutions of the Laplace equation in prolate spheroidal coordinates are the well known *prolate spheroidal harmonics* [14,25,30]. The prolate spheroidal harmonics form a complete and orthogomal set of functions for the space interior of (2.2), and are a combination of products of spherical functions:

$$\Theta(\theta)\ \Xi(\mu) \begin{cases} T_l(\cos\phi) \\ \\ \sin\phi\, U_{l-1}(\cos\phi) \end{cases}, \quad 0 \le l \le n, \tag{2.3}$$

where n is a constant, and $\Theta(\theta) := P_n^l(\cos\theta)$ and $\Xi(\mu) := P_n^l(\cosh\mu)$ satisfy the differential equations

$$\frac{d^2\Theta(\theta)}{d\theta^2} + \cot\theta \frac{d\Theta(\theta)}{d\theta} + \Big[n(n+1) - \frac{l^2}{\sin^2\theta}\Big]\Theta(\theta)\sin\theta = 0,$$

$$\frac{d^2\Xi(\mu)}{d\mu^2} + \coth\mu \frac{d\Xi(\mu)}{d\mu} - \Big[\frac{l^2}{\sinh^2\mu} + n(n+1)\Big]\Xi(\mu)\sinh\mu = 0.$$

Here P_n^l are the *Ferrer's associated Legendre functions* of the first kind of n-th degree and l-th order, T_l and U_l are, respectively, the *Chebyshev polynomials* of the first and second kinds that satisfy the following three-term recurrence relations:

$$T_{l+1}(s) = 2s\, T_l(s) - T_{l-1}(s),$$
$$U_{l+1}(s) = 2s\, U_l(s) - U_{l-1}(s), \qquad l = 1, 2, \ldots,$$

for real argument $s = \cos\phi$ with starting values $T_0(s) = U_0(s) = 1$, and $2T_1(s) = U_1(s) = 2s$. Special values of T_l and U_l which are easily derived from the above relations are

$$T_l(1) = 1, \qquad T_l(-1) = (-1)^l;$$
$$U_l(1) = l + 1, \qquad U_l(-1) = (l+1)(-1)^l.$$

We remark that whenever $l = 0$, the corresponding associated Legendre function P_n^0 coincides with the Legendre polynomial P_n. In this assignment, the sign convention of including the Condon-Shortley phase is adopted:

$$P_n^l(\cosh\mu) := (-1)^l (\sinh\mu)^l \, \frac{d^l}{dt^l}[P_n(t)]\bigg|_{t=\cosh\mu}.$$

Originally, the spheroidal functions were introduced by C. Niven in 1880 while studying the conduction of heat in an ellipsoid of revolution, which lead to a Helmholtz equation in spheroidal coordinates. The prolate spheroidal harmonics are special functions in mathematical physics which have found many important practical applications in science and engineering where the spheroidal coordinate system is used. They usually appear in the solutions of Dirichlet problems in spheroidal domains arising in hydrodynamics, elasticity and electromagnetism. For the solvability of boundary value problems of radiation, scattering, and propagation of acoustic signals and electromagnetism waves in spheroidal structures, spheroidal wave functions are commonly encountered. Contrary to the classical spherical harmonics, there are a few references concerning the numerical evaluation of prolate spheroidal harmonics [53]. Recently, there has been a growing interest in developing numerical methods using prolate spheroidal functions as basis functions, see [4,8,53,54,55], among others. These applications have stimulated a surge of new techniques and have reawakened interest in approximation theory, potential theory, and the theory of partial differential equations of elliptic type for spheroidal domains.

Higher dimensional extensions of the prolate spheroidal functions were first studied by D. Slepian in [49], which provided many of their analytical properties, as well as properties that support the construction of numerical schemes (see also S. Pei et al. [44], and A. Zayed [56]). Very recently, K. Kou et al. [26] introduced the continuous Clifford prolate spheroidal functions in the finite Clifford Fourier transform setting. These generalized spheroidal functions were successfully applied for the analysis of the energy concentration problem introduced in the early-sixties by D. Slepian and H. Pollak [48].

2.2. *The Riesz and Moisil-Théodoresco systems*

As is well known, a holomorphic function $f(z) = u(x,y) + iv(x,y)$ defined in an open domain of the complex plane, satisfies the Cauchy-Riemann system

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}.$$

As in the case of two variables, we may now characterize two possible analogues of the Cauchy-Riemann system in an open domain of the Euclidean space $\mathbb{R}^3$. More precisely, consider the pair $f = (f_0, f^*)$ where f_0 is a real-valued continuously differentiable function defined on an open domain $\Omega \subset \mathbb{R}^3$, and $f^* = (f_1, f_2, f_3)$ is a continuously differentiable vector-field in Ω for which

$$\text{(R)} \quad \begin{cases} \text{div } f^* = 0 \\ \text{rot } f^* = 0 \end{cases}. \tag{2.4}$$

Recall that the 3-tuple f^* is said to be an *M. Riesz system of conjugate harmonic functions* in the sense of E. Stein and G. Weiß [50,51], and system (R) is called the *Riesz system* [45]. The Riesz system has a physical relevance as it describes the velocity field of a stationary flow of a non-compressible fluid without sources nor sinks.

The *Moisil-Théodoresco system* is represented by [34] (cf. [27])

$$\text{(MT)} \quad \begin{cases} \text{div } f^* = 0 \\ \text{grad } f_0 + \text{rot } f^* = 0 \end{cases}, \tag{2.5}$$

and it is closely related to many mathematical models of relevance in spatial physical problems such as the Lamé system [20], and Stokes system [21] (see also A. Bitsadze [2], and A. Dzhuraev [13]). Both systems are historical precursors that generalize the classical Cauchy-Riemann system in the plane. Obviously (2.4) may be derived from (2.5) by taking $f_0 = 0$.

2.3. *Quaternion analysis*

In the present section, we begin by reviewing some definitions and basic properties of quaternion analysis. For all what follows we will work in $\mathbb{H}$, the skew field of quaternions. This means we can express each element $\mathbf{z} \in \mathbb{H}$ uniquely in the form

$$\mathbf{z} = z_0 + z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k}, \quad z_l \in \mathbb{R} \ \ (l = 0,1,2,3)$$

where the imaginary units $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ stand for the elements of the basis of $\mathbb{H}$, and obey the following laws of multiplication:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1;$$

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \qquad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \qquad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

The scalar and vector parts of $\mathbf{z}$, $\mathrm{Sc}(\mathbf{z})$ and $\mathrm{Vec}(\mathbf{z})$, are defined as the z_0 and $z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k}$ terms, respectively. Like in the complex case, the conjugate of $\mathbf{z}$ is the quaternion $\overline{\mathbf{z}} = z_0 - z_1\mathbf{i} - z_2\mathbf{j} - z_3\mathbf{k}$, and the norm $|\mathbf{z}|$ of $\mathbf{z}$ is defined by

$$|\mathbf{z}| = \sqrt{\mathbf{z}\overline{\mathbf{z}}} = \sqrt{\overline{\mathbf{z}}\mathbf{z}} = \sqrt{z_0^2 + z_1^2 + z_2^2 + z_3^2}.$$

We shall always assume the quaternion $0 + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} := \mathbf{0}_{\mathbb{H}}$ to be the neutral element of addition in the sequel.

The *paravector space* is the linear subspace defined by

$$\mathcal{A} := \mathrm{span}_{\mathbb{R}}\{1, \mathbf{i}, \mathbf{j}\} \subset \mathbb{H},$$

with elements of the form $\mathbf{x} := x_0 + x_1\mathbf{i} + x_2\mathbf{j}$. As a matter of fact, throughout the text we will often use the symbol x to represent a point in $\mathbb{R}^3$ and $\mathbf{x}$ to represent the corresponding paravector. Of course, it is assumed here that $\mathcal{A}$ is a real vectorial subspace, but not a subalgebra of $\mathbb{H}$.

Let us make a notation convention. We say that

$$\mathbf{f} : \mathcal{S} \longrightarrow \mathbb{H}, \qquad \mathbf{f}(x) := [\mathbf{f}(x)]_0 + [\mathbf{f}(x)]_1\mathbf{i} + [\mathbf{f}(x)]_2\mathbf{j} + [\mathbf{f}(x)]_3\mathbf{k} \tag{2.6}$$

is a quaternion-valued function or, briefly, an $\mathbb{H}$-valued function, where the components $[\mathbf{f}]_l$ $(l = 0, 1, 2, 3)$ are real-valued functions defined in $\mathcal{S}$. By now, it is clear that the form of a paravector-valued function may be derived from (2.6) by taking $[\mathbf{f}(x)]_3 = 0$. Continuity, differentiability, integrability, and so on, which are ascribed to $\mathbf{f}$ are defined componentwise. We will work with both the real- (resp. quaternionic-) linear Hilbert space of square integrable $\mathcal{A}$- (resp. $\mathbb{H}$-) valued functions defined in $\mathcal{S}$, that we denote by $L^2(\mathcal{S}; \mathcal{A}; \mathbb{R})$ (resp. $L^2(\mathcal{S}; \mathbb{H}; \mathbb{H})$). In this assignment, the scalar and quaternionic inner products are defined by

$$< \mathbf{f}, \mathbf{g} >_{L^2(\mathcal{S};\mathcal{A};\mathbb{R})} := \int_{\mathcal{S}} \mathrm{Sc}(\overline{\mathbf{f}}\,\mathbf{g})\, dV \tag{2.7}$$

and

$$< \mathbf{f}, \mathbf{g} >_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})} := \int_{\mathcal{S}} \overline{\mathbf{f}}\,\mathbf{g}\, dV\,, \tag{2.8}$$

where dV denotes the volume of $\mathcal{S}$ normalized so that $V(\mathcal{S}) = 1$.

For continuously real-differentiable $\mathcal{A}$-valued functions $\mathbf{f}$, the reader may be familiar with the (reduced) quaternionic operator

$$D = \frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2},$$

which is called generalized Cauchy-Riemann operator on $\mathbb{R}^3$. From this operator we obtain the usual Dirac operator

$$\partial = \mathbf{i}\partial_{y_1} + \mathbf{j}\partial_{y_2} + \mathbf{k}\partial_{y_3}$$

via the equality $\partial = -\mathbf{j}D\mathbf{i}$, and the identification

$$\mathbf{x} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} \in \mathcal{A} \quad \rightarrow \quad \mathbf{y} = x_2\mathbf{i} + x_1\mathbf{j} + x_0\mathbf{k} \in \mathbb{H}.$$

Namely, a continuously real-differentiable $\mathcal{A}$-valued function $\mathbf{f}$ is said to be *monogenic* in $\mathcal{S}$ if $D\mathbf{f} = 0 = \mathbf{f}D$ in $\mathcal{S}$, which is equivalent to the Riesz system

$$\text{(R)} \quad \begin{cases} \dfrac{\partial[\mathbf{f}]_0}{\partial x_0} - \dfrac{\partial[\mathbf{f}]_1}{\partial x_1} - \dfrac{\partial[\mathbf{f}]_2}{\partial x_2} = 0\,, \\[2ex] \dfrac{\partial[\mathbf{f}]_0}{\partial x_1} + \dfrac{\partial[\mathbf{f}]_1}{\partial x_0} = 0, \quad \dfrac{\partial[\mathbf{f}]_0}{\partial x_2} + \dfrac{\partial[\mathbf{f}]_2}{\partial x_0} = 0, \quad \dfrac{\partial[\mathbf{f}]_1}{\partial x_2} - \dfrac{\partial[\mathbf{f}]_2}{\partial x_1} = 0\,. \end{cases}$$

This system can also be written in abbreviated form:

$$\begin{cases} \operatorname{div} \overline{\mathbf{f}} \ = 0 \\ \operatorname{curl} \overline{\mathbf{f}} = 0 \end{cases}.$$

For the interpretation of the (R)-system in viewpoint of $\mathbb{H} \cong \mathcal{C}\ell_{0,3}^{+}$ we refer to [11]. Following [31], the solutions of the system (R) are customary called (R)-solutions. The subspace of polynomial (R)-solutions of degree n will be denoted by $\mathcal{R}^{+}(\mathcal{S};\mathcal{A};n)$. We also denote by $\mathcal{R}^{+}(\mathcal{S};\mathcal{A}) := L^2(\mathcal{S};\mathcal{A};\mathbb{R}) \cap \ker D$ the space of square integrable $\mathcal{A}$-valued monogenic functions defined in $\mathcal{S}$.

The analysis of functions with values in $\mathbb{H}$ requires a different treatment. Namely, an $\mathbb{H}$-valued function $\mathbf{f}$ is called *left* (resp. *right*) monogenic in $\mathcal{S}$ if $\mathbf{f}$ is in $C^1(\mathcal{S};\mathbb{H})$ and satisfies $\partial\mathbf{f} = 0$ (resp. $\mathbf{f}\partial = 0$) in $\mathcal{S}$. Throughout the text we only use left $\mathbb{H}$-valued monogenic functions that, for simplicity, we call monogenic. Nevertheless, all results accomplished to left $\mathbb{H}$-valued monogenic functions can be easily adapted to right $\mathbb{H}$-valued monogenic functions.

For any $\mathbb{H}$-valued function $\mathbf{f}$ it is worthy of note that the equation $\partial\mathbf{f} = 0$ is equivalent to the system

$$\text{(MT)} \quad \begin{cases} \dfrac{\partial[\mathbf{f}]_1}{\partial x_0} + \dfrac{\partial[\mathbf{f}]_2}{\partial x_1} + \dfrac{\partial[\mathbf{f}]_3}{\partial x_2} &= 0 \\[2ex] \dfrac{\partial[\mathbf{f}]_0}{\partial x_0} - \dfrac{\partial[\mathbf{f}]_2}{\partial x_2} + \dfrac{\partial[\mathbf{f}]_3}{\partial x_1} &= 0 \\[2ex] \dfrac{\partial[\mathbf{f}]_0}{\partial x_1} + \dfrac{\partial[\mathbf{f}]_1}{\partial x_2} - \dfrac{\partial[\mathbf{f}]_3}{\partial x_0} &= 0 \\[2ex] \dfrac{\partial[\mathbf{f}]_0}{\partial x_2} - \dfrac{\partial[\mathbf{f}]_1}{\partial x_1} + \dfrac{\partial[\mathbf{f}]_2}{\partial x_0} &= 0 \end{cases}$$

or, in a more compact form:

$$\begin{cases} \operatorname{div}(\operatorname{Vec}(\mathbf{f})) = 0 \\ \operatorname{grad}[\mathbf{f}]_0 + \operatorname{rot}(\operatorname{Vec}(\mathbf{f})) = 0. \end{cases}$$

For the interpretation of the (MT) system in viewpoint of $\mathbb{H} \cong \mathcal{C}\ell_{0,3}^{+}$ we also refer to [12]. To state our general results we shall need some further notation. The solutions of the (MT)-system are called (MT)-solutions, and the subspace of polynomial (MT)-solutions of degree n is denoted by $\mathcal{M}^{+}(\mathcal{S};\mathbb{H};n)$. In [52] (see also [12]), it is proved that $\dim \mathcal{M}^{+}(\mathcal{S};\mathbb{H};n) = n+1$. We also denote by $\mathcal{M}^{+}(\mathcal{S};\mathbb{H}) := L^2(\mathcal{S};\mathbb{H};\mathbb{H}) \cap \ker \partial$ the space of square integrable $\mathbb{H}$-valued monogenic functions defined in $\mathcal{S}$.

3. Complete orthogonal systems of monogenic polynomials over 3D prolate spheroids

3.1. *3D Prolate spheroidal monogenics - Riesz system*

In [39,40] (cf. [42]) a special complete orthogonal system of (R)-polynomial solutions has been obtained over the interior of 3D prolate spheroids, which can be seen as a refinement of the harmonic polynomial systems exploited by P. Garabedian in [18]. As a first step towards the orthogonality of the polynomials in question does not depend on the shape of the spheroids, but only on the location of the foci of the ellipse generating the spheroid.

The explicit expressions of the mentioned spheroidal monogenics are

Theorem 3.1. *Monogenic polynomials of the form*

$$\begin{aligned}
&\boldsymbol{\mathcal{E}}_{n,l}\,(\mu,\theta,\phi) := \frac{(n+l+1)}{2}\, A_{n,l}(\mu,\theta)\, T_l(\cos\phi) \\
&+ \frac{1}{4\,(n-l+1)}\, A_{n,l+1}(\mu,\theta)\, [T_{l+1}(\cos\phi)\mathbf{i} + \sin\phi\, U_l(\cos\phi)\mathbf{j}] \\
&+ \frac{1}{4}\,(n+1+l)(n+l)(n-l+2)\, A_{n,l-1}(\mu,\theta)\, [-T_{l-1}(\cos\phi)\mathbf{i} + \sin\phi\, U_{l-2}(\cos\phi)\mathbf{j}]\,,
\end{aligned}$$

and

$$\begin{aligned}
&\boldsymbol{\mathcal{F}}_{n,m}\,(\mu,\theta,\phi) := \frac{(n+m+1)}{2}\, A_{n,m}(\mu,\theta)\sin\phi\, U_{m-1}(\cos\phi) \\
&+ \frac{1}{4\,(n-m+1)}\, A_{n,m+1}(\mu,\theta)\, [\sin\phi\, U_m(\cos\phi)\mathbf{i} - T_{m+1}(\cos\phi)\mathbf{j}] \\
&- \frac{1}{4}\,(n+1+m)(n+m)(n-m+2)\, A_{n,m-1}(\mu,\theta)\, [\sin\phi\, U_{m-2}(\cos\phi)\mathbf{i} + T_{m-1}(\cos\phi)\mathbf{j}]\,,
\end{aligned}$$

for $l = 0, \ldots, n+1$ and $m = 1, \ldots, n+1$, with the notation

$$A_{n,l}(\mu,\theta) := \sum_{k=0}^{\lceil \frac{n-l}{2} \rceil} \frac{(2n+1-2k)\,(n+l)_{2k}}{(n+1-l)_{2k+1}}\, P^l_{n-2k}(\cosh\mu)\, P^l_{n-2k}(\cos\theta) \qquad (3.1)$$

such that $A_{n,-1} = -\frac{1}{n(n+1)^2(n+2)} A_{n,1}$ form a complete orthogonal system for the interior of the prolate spheroid (2.2) in the sense of the scalar product (2.7).

Remark 3.1. In [39] it is shown a corresponding orthogonality over the surface $\partial\mathcal{S}$ of these spheroids in the sense of the scalar product

$$\int_{\partial\mathcal{S}} \mathrm{Sc}(\overline{\mathbf{f}}\,\mathbf{g})\,\boldsymbol{\omega}\,d\sigma$$

where $d\sigma$ denotes the Lebesgue measure on $\partial\mathcal{S}$, and with weight function

$$\boldsymbol{\omega} := |c^2 - (ca\cos\theta + \mathbf{i}\,cb\sin\theta)^2|^{1/2}(\sin^2\theta + \sinh^2\mu) \quad (a > b). \qquad (3.2)$$

This function equals to the square root of the product of the distances from any point inside of the spheroid to the north and south poles, $(c,0,0)$ and $(-c,0,0)$, respectively.

Remark 3.2. For the usual applications we define the $2n+3$ spheroidal monogenics $\boldsymbol{\mathcal{E}}_{n,l}$ $(l = 0, \ldots, n+1)$ and $\boldsymbol{\mathcal{F}}_{n,m}$ $(m = 1, \ldots, n+1)$ in a spheroid which has an infinite boundary, because $P_n^l(\cosh\mu)$ becomes infinite with μ. Of course, the results can be extended to the case of the region outside a spheroid as well. One has merely to replace the Ferrer's associated Legendre functions by the Legendre functions of second kind [25].

Next we summarize some basic properties of the polynomials, which illustrate the important role that they play in the theory of monogenic functions [39].

Theorem 3.2. *The spheroidal monogenics $\boldsymbol{\mathcal{E}}_{n,l}$ and $\boldsymbol{\mathcal{F}}_{n,m}$ satisfy the following properties:*

(1) $\boldsymbol{\mathcal{E}}_{n,l}$ and $\boldsymbol{\mathcal{F}}_{n,m}$ are the zero functions for $l, m \geq n+2$;

(2) $\boldsymbol{\mathcal{E}}_{n,l}$ and $\boldsymbol{\mathcal{F}}_{n,m}$ are 2π-periodic with respect to the variable ϕ;

(3) For each $n \in \mathbb{N}_0$, the harmonic polynomials $\mathrm{Sc}(\boldsymbol{\mathcal{E}}_{n,l})$ $(l = 0, \ldots, n)$ and $\mathrm{Sc}(\boldsymbol{\mathcal{F}}_{n,m})$ $(m = 1, \ldots, n)$ form a complete orthogonal system for the interior of (2.2) in the sense of the scalar product (2.7);

(4) For each $n \in \mathbb{N}_0$, the harmonic polynomials in each of the two sets $\{\mathrm{Sc}(\boldsymbol{\mathcal{E}}_{n,l}), [\boldsymbol{\mathcal{E}}_{n,l}]_1, [\boldsymbol{\mathcal{E}}_{n,l}]_2 : l = 0, \ldots, n+1\}$, and $\{\mathrm{Sc}(\boldsymbol{\mathcal{F}}_{n,m}), [\boldsymbol{\mathcal{F}}_{n,m}]_1, [\boldsymbol{\mathcal{F}}_{n,m}]_2 : m = 1, \ldots, n+1\}$ are orthogonal for the interior of (2.2) in the sense of the scalar product (2.7).

Special values of the spheroidal monogenics $\boldsymbol{\mathcal{E}}_{n,l}$ and $\boldsymbol{\mathcal{F}}_{n,m}$ are:

$$\begin{aligned}\boldsymbol{\mathcal{E}}_{n,l}(\mu,\theta,0) = \frac{(n+l+1)}{2} A_{n,l}(\mu,\theta) + \frac{1}{4}\Big[(n-l+1)A_{n,l+1}(\mu,\theta) \\ - (n+l+1)(n+l)(n-l+2)A_{n,l-1}(\mu,\theta)\Big]\mathbf{i};\end{aligned}$$

$$\begin{aligned}\boldsymbol{\mathcal{E}}_{n,l}(\mu,\theta,\pi) = \frac{(n+l+1)}{2} A_{n,l}(\mu,\theta)(-1)^l - \frac{1}{4}(-1)^l\Big[(n-l+1)A_{n,l+1}(\mu,\theta) \\ - (n+l+1)(n+l)(n-l+2)A_{n,l-1}(\mu,\theta)\Big]\mathbf{i};\end{aligned}$$

$$\begin{aligned}\boldsymbol{\mathcal{F}}_{n,m}(\mu,\theta,0) = -\frac{1}{4}\Big[(n-m+1)A_{n,m+1}(\mu,\theta) \\ + (n+1+m)(n+m)(n-m+2)A_{n,m-1}(\mu,\theta)\Big]\mathbf{j};\end{aligned}$$

$$\begin{aligned}\boldsymbol{\mathcal{F}}_{n,m}(\mu,\theta,\pi) = (-1)^m\frac{1}{4}\Big[(n-m+1)A_{n,m+1}(\mu,\theta) \\ + (n+1+m)(n+m)(n-m+2)A_{n,m-1}(\mu,\theta)\Big]\mathbf{j}.\end{aligned}$$

3.2. *Recurrence relations*

In view of the practical applicability of the aforementioned spheroidal monogenics, next we illustrate explicit recurrence rules between them as a consequence of Theorem 3.1 that are plain to be integrated in a computational framework.

Proposition 3.1. *For each $n \in \mathbb{N}_0$, $\boldsymbol{\mathcal{E}}_{n,l}$ and $\boldsymbol{\mathcal{F}}_{n,m}$ satisfy the recurrence formulae:*

$$\begin{aligned}\boldsymbol{\mathcal{E}}_{n,l} = &-\frac{(n+l+1)(n-l+2)}{2}\left(\boldsymbol{\mathcal{E}}_{n,l-1}\mathbf{i} - \boldsymbol{\mathcal{F}}_{n,l-1}\mathbf{j}\right)\\ &+\frac{(1+\delta_{0,l})}{2\,(n+l+2)(n-l+1)}\left(\boldsymbol{\mathcal{E}}_{n,l+1}\mathbf{i} + \boldsymbol{\mathcal{F}}_{n,l+1}\mathbf{j}\right);\end{aligned}$$

$$\begin{aligned}\boldsymbol{\mathcal{F}}_{n,m} = &-\frac{(n+m+1)(n-m+2)}{2}\left(\boldsymbol{\mathcal{F}}_{n,m-1}\mathbf{i} + \boldsymbol{\mathcal{E}}_{n,m-1}\mathbf{j}\right)\\ &+\frac{1}{2\,(n+m+2)(n-m+1)}\left(\boldsymbol{\mathcal{F}}_{n,m+1}\mathbf{i} - \boldsymbol{\mathcal{E}}_{n,m+1}\mathbf{j}\right)\end{aligned}$$

for $l = 0, \ldots, n+1$ and $m = 1, \ldots, n+1$, with starting values $\boldsymbol{\mathcal{E}}_{0,0} = \boldsymbol{\mathcal{E}}_{0,1}\mathbf{i} = \boldsymbol{\mathcal{F}}_{0,1}\mathbf{j}$. For a more unified formulation we remind the reader that $\boldsymbol{\mathcal{E}}_{n,m} = \boldsymbol{\mathcal{F}}_{n,m} \equiv 0$ for $m \geq n+2$, and $\delta_{l,0}$ is the Kronecker symbol.

Proof. For simplicity we just present the calculations for $\boldsymbol{\mathcal{E}}_{n,m}$ $(m > 1)$. By direct inspection of previous expressions one has

$$\begin{aligned}&-\frac{(n+m+1)(n-m+2)}{2}\left(\boldsymbol{\mathcal{E}}_{n,m-1}\mathbf{i} - \boldsymbol{\mathcal{F}}_{n,m-1}\mathbf{j}\right)\\ &\qquad\qquad+\frac{1}{2\,(n+m+2)(n-m+1)}\left(\boldsymbol{\mathcal{E}}_{n,m+1}\mathbf{i} + \boldsymbol{\mathcal{F}}_{n,m+1}\mathbf{j}\right)\\ &= \frac{(n+m+1)}{2}\,A_{n,m}(\mu,\theta)\,T_m(\cos\phi)\\ &+ \frac{1}{4\,(n-m+1)}\,A_{n,m+1}(\mu,\theta)\,T_{m+1}(\cos\phi)\mathbf{i}\\ &- \frac{1}{4}\,(n+1+m)(n+m)(n-m+2)\,A_{n,m-1}(\mu,\theta)\,T_{m-1}(\cos\phi)\mathbf{i}\\ &+ \frac{1}{4}\,(n+1+m)(n+m)(n-m+2)\,A_{n,m-1}(\mu,\theta)\sin\phi\,U_{m-2}(\cos\phi)\mathbf{j}\\ &+ \frac{1}{4\,(n-m+1)}\,A_{n,m+1}(\mu,\theta)\,\sin\phi\,U_m(\cos\phi)\mathbf{j}\\ &:= \boldsymbol{\mathcal{E}}_{n,m}, \quad m = 1, \ldots, n+1.\end{aligned}$$

The proofs for $\boldsymbol{\mathcal{E}}_{n,0}$ and $\boldsymbol{\mathcal{F}}_{n,m}$ $(m > 1)$ follow the same principle and are therefore straightforward. □

Remark 3.3. Under these conditions, the previous recurrence relations allow to find all the polynomials, since for each degree n it is easy to find the initial values $\boldsymbol{\mathcal{E}}_{0,1}$ and $\boldsymbol{\mathcal{F}}_{0,1}$ directly from the generating function $\boldsymbol{\mathcal{E}}_{0,0}$.

As a direct consequence, we obtain the two-term type recurrence relation.

Corollary 3.1. *For each $n \in \mathbb{N}_0$, $\boldsymbol{\mathcal{E}}_{n,l}$ and $\boldsymbol{\mathcal{F}}_{n,m}$ satisfy the two-term type recurrence formula:*

$$
\begin{aligned}
&(n+2)(n+1)\,\boldsymbol{\mathcal{E}}_{n,0} - \boldsymbol{\mathcal{E}}_{n,1}\mathbf{i} - \boldsymbol{\mathcal{F}}_{n,1}\mathbf{j} \;=\; 0,\\
&(n+m+1)(n-m+2)\,(\boldsymbol{\mathcal{E}}_{n,m-1} - \boldsymbol{\mathcal{F}}_{n,m-1}\mathbf{k}) - \boldsymbol{\mathcal{E}}_{n,m}\mathbf{i} - \boldsymbol{\mathcal{F}}_{n,m}\mathbf{j} \;=\; 0
\end{aligned}
$$

for $m = 1, \ldots, n+1$, with the starting value $\boldsymbol{\mathcal{E}}_{0,0} = \frac{1}{2}$.

3.3. *3D Prolate spheroidal monogenics - Moisil Théodoresco system*

This subsection extends the above results to a quaternionic Hilbert subspace; in particular, we exploit a complete orthogonal system of (MT)-polynomial solutions over 3D prolate spheroids. For more details we refer the reader to [40].

Following [40] we designate the new $n+1$ (prolate) spheroidal monogenics by

$$\boldsymbol{\mathcal{S}}_{n,l} := \boldsymbol{\mathcal{E}}_{n,l+1}\,\mathbf{i} + \boldsymbol{\mathcal{F}}_{n,l+1}\,\mathbf{j}, \quad l = 0, \ldots, n, \tag{3.3}$$

namely functions with respect to the variables μ, θ, and the azimuthal angle ϕ of the quaternion form:

$$
\begin{aligned}
\boldsymbol{\mathcal{S}}_{n,l}\,(\mu,\theta,\phi) :=\; & \frac{1}{2}\,(n+2+l)\,(n+1+l)\,(n-l+1)\,A_{n,l}(\mu,\theta)\,T_l(\cos\phi)\\
+\; & \frac{1}{2}\,(n+2+l)\,A_{n,l+1}(\mu,\theta)\,T_{l+1}(\cos\phi)\,\mathbf{i}\\
+\; & \frac{1}{2}\,(n+2+l)\,A_{n,l+1}(\mu,\theta)\,\sin\phi\,U_l(\cos\phi)\,\mathbf{j}\\
-\; & \frac{1}{2}\,(n+2+l)\,(n+1+l)\,(n-l+1)\,A_{n,l}(\mu,\theta)\,\sin\phi\,U_{l-1}(\cos\phi)\,\mathbf{k}
\end{aligned}
$$

with the subscript coefficient function $A_{n,l}(\mu,\theta)$ given by (3.1). It is easily verified that the polynomials $\boldsymbol{\mathcal{S}}_{n,l}$ are the zero functions for $l \geq n+1$.

Remark 3.4. Although the spheroidal monogenics $\boldsymbol{\mathcal{S}}_{n,0}$ are built in terms of the Legendre polynomials while $\boldsymbol{\mathcal{S}}_{n,l}$ ($l > 0$) are built in terms of the Ferrer's associated Legendre functions, we still include the treatment of the first into the general case, whenever this does not raise any confusion and the treatment remains the same.

These $n+1$ polynomials satisfy the first order partial differential equation

$$
\begin{aligned}
0 &= c\,\partial\boldsymbol{\mathcal{S}}_{n,l}\\
&= \mathbf{i}\left(\frac{\cos\theta\sinh\mu}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu} - \frac{\sin\theta\cosh\mu}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta}\right)\\
&+ \mathbf{j}\left(\frac{\sin\theta\cosh\mu\cos\phi}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu} + \frac{\cos\theta\sinh\mu\cos\phi}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta} - \frac{\sin\phi}{\sin\theta\sinh\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\phi}\right)\\
&+ \mathbf{k}\left(\frac{\sin\theta\cosh\mu\sin\phi}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu} + \frac{\cos\theta\sinh\mu\sin\phi}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta} + \frac{\cos\phi}{\sin\theta\sinh\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\phi}\right).
\end{aligned}
$$

We further assume the reader to be familiar with the fact that ∂ is a square root of the Laplace operator in $\mathbb{R}^3$ in the sense that

$$\begin{aligned}\Delta_3 \boldsymbol{\mathcal{S}}_{n,l} &= -\partial^2 \boldsymbol{\mathcal{S}}_{n,l} \\ &= \frac{1}{c^2(\sin^2\theta + \sinh^2\mu)}\left(\frac{\partial^2 \boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu^2} + \frac{\partial^2 \boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta^2} + \coth\mu \frac{\partial \boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu} + \cot\theta \frac{\partial \boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta}\right) \\ &+ \frac{1}{c^2 \sin^2\theta \sinh^2\mu}\frac{\partial^2 \boldsymbol{\mathcal{S}}_{n,l}}{\partial\phi^2}.\end{aligned}$$

It is of interest to remark at this point that the Laplacian in (prolate) spheroidal coordinates reduces to the classical Laplacian in spherical coordinates if $a = b$, which occurs as μ appoaches infinity, and in which case the two foci coincide at the origin.

In [40] it is shown that the above-mentioned polynomails are (MT)-solutions and form a complete orthogonal system over the interior of 3D prolate spheroids.

Theorem 3.3. *For each $n \in \mathbb{N}_0$, the set $\{\boldsymbol{\mathcal{S}}_{n,l} : l = 0, \ldots, n\}$ is orthogonal over the interior of the prolate spheroid (2.2) in the sense of the quaternion product (2.8), and their norms are given by*

$$\begin{aligned}\|\boldsymbol{\mathcal{S}}_{n,l}\|^2_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})} &= \pi\,(n+2+l)\,\frac{(n+2+l)!}{(n-l)!} \\ \times \Bigg[(n+1+l)(n-l+1)&\int_0^\alpha P_n^l(\cosh\mu)\,\sinh\mu\,\cosh\mu\,P_{n+1}^l(\cosh\mu)\,d\mu \\ &- \frac{(n+1+l)^2(n-l+1)}{2n+3}\int_0^\alpha [P_n^l(\cosh\mu)]^2\,\sinh\mu\,d\mu \\ &+ \int_0^\alpha P_n^{l+1}(\cosh\mu)\,\sinh\mu\,\cosh\mu\,P_{n+1}^{l+1}(\cosh\mu)\,d\mu \\ &- \frac{(n+2+l)}{2n+3}\int_0^\alpha [P_n^{l+1}(\cosh\mu)]^2\,\sinh\mu\,d\mu\Bigg].\end{aligned}$$

4. Approximation of monogenic functions in 3D prolate spheroids

4.1. *A special Fourier expansion by means of prolate spheroidal monogenics*

This subsection discusses a suitable Fourier expansion for monogenic functions over 3D prolate spheroids in terms of orthogonal monogenic polynomials. To begin with, note that for each degree $n \in \mathbb{N}_0$ the set

$$\{\boldsymbol{\mathcal{S}}_{n,l} : l = 0, \ldots, n\} \tag{4.1}$$

is formed by $n+1 = \dim \mathcal{M}^+(\mathcal{S};\mathbb{H};n)$ monogenic polynomials, and therefore, it is complete in $\mathcal{M}^+(\mathcal{S};\mathbb{H};n)$. Based on the orthogonal decomposition

$$\mathcal{M}^+(\mathcal{S};\mathbb{H}) = \oplus\sum_{n=0}^{\infty} \mathcal{M}^+(\mathcal{S};\mathbb{H};n),$$

and the completeness of the system in each subspace $\mathcal{M}^+(\mathcal{S};\mathbb{H};n)$, it follows the result [40].

Theorem 4.1. *For each n the set (4.1) forms an orthogonal basis in the subspace $\mathcal{M}^+(\mathcal{S};\mathbb{H};n)$ in the sense of the product (2.8). Consequently,*

$$\{\boldsymbol{\mathcal{S}}_{n,l} : l = 0,\dots,n;\ n = 0,1,\dots\} \tag{4.2}$$

is an orthogonal basis in $\mathcal{M}^+(\mathcal{S};\mathbb{H})$.

Using the L^2-norms of the constructed polynomials, we can easily normalize them in order to get a complete orthonormal system in $L^2(\mathcal{S};\mathbb{H};\mathbb{H})$. For, from now on we shall denote by $\boldsymbol{\mathcal{S}}^*_{n,l}$ $(l = 0,\dots,n)$ the new normalized basis functions $\boldsymbol{\mathcal{S}}_{n,l}$ in $L^2(\mathcal{S};\mathbb{H};\mathbb{H})$.

Yet clearly we can easily write down the Fourier expansion of a square integrable $\mathbb{H}$-valued monogenic function in 3D prolate spheroids.

Lemma 4.1 (Fourier Expansion). *Let $\mathbf{f} \in \mathcal{M}^+(\mathcal{S};\mathbb{H})$. The function $\mathbf{f}$ can be represented with the orthogonal system (4.2):*

$$S(\mathbf{f})(x) = \sum_{n=0}^{\infty}\sum_{l=0}^{n} \boldsymbol{\mathcal{S}}_{n,l}\, a^*_{n,l}, \tag{4.3}$$

where for each $n \in \mathbb{N}_0$, the associated (quaternion-valued) Fourier coefficients are given by

$$a^*_{n,l} = \frac{< \mathbf{f}, \boldsymbol{\mathcal{S}}_{n,l} >_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})}}{||\boldsymbol{\mathcal{S}}_{n,l}||^2_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})}}.$$

To proceed with, we state some definitions that will be needed in the sequel.

Definition 4.1. *Let k be a fixed natural number. For each n, let*

$$\mathbf{G}_k(x,y) := \sum_{n=0}^{k}\sum_{l=0}^{n} \frac{\overline{\boldsymbol{\mathcal{S}}_{n,l}(y)}\,\boldsymbol{\mathcal{S}}_{n,l}(x)}{2\|\boldsymbol{\mathcal{S}}_{n,l}\|^2_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})}} \tag{4.4}$$

for all $x, y \in \mathcal{S}$.

Definition 4.2. *Let k be a fixed natural number. For each n, let*

$$\mathbf{P}_k(\mathbf{f})(x) := \sum_{n=0}^{k}\sum_{l=0}^{n} \boldsymbol{\mathcal{S}}^{n,l}(\mathbf{f})(x) \tag{4.5}$$

where

$$\boldsymbol{\mathcal{S}}^{n,l}(\mathbf{f})(x) := \frac{\boldsymbol{\mathcal{S}}_{n,l}(x)}{2\|\boldsymbol{\mathcal{S}}_{n,l}\|^2_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})}} \int_{\mathcal{S}} \overline{\mathbf{f}(y)}\boldsymbol{\mathcal{S}}_{n,l}(y)\, dV(y),$$

for all $x, y \in \mathcal{S}$.

We can now prove the main result of this subsection.

Proposition 4.1. *Let k be a fixed natural number, and let*

$$\mathbf{s}_k(\mathbf{f})(x) = \sum_{n=0}^{k} \sum_{l=0}^{n} \boldsymbol{\mathcal{S}}_{n,l}(x)\, a^*_{n,l},$$

for all $x \in \mathcal{S}$. Assume there exist the limits

$$\mathbf{P}_k(\mathbf{f}) \longrightarrow \mathbf{P}(\mathbf{f}),$$

and

$$\int_{\mathcal{S}} \overline{\mathbf{G}_k(y)}\mathbf{f}(y)dV(y) \longrightarrow \mathbf{G}(\mathbf{f}),$$

in the L^2 sense as n approaches infinity, where $\mathbf{G}_k$ and $\mathbf{P}_k$ are given by (4.4) and (4.5). It holds

$$\mathbf{s}_k(\mathbf{f}) \longrightarrow_{n \longrightarrow \infty} \mathbf{P}(\mathbf{f}) + \mathbf{G}(\mathbf{f})$$

in the L^2 sense.

Proof. Let k be a fixed natural number, and $x \in \mathcal{S}$. A direct computation shows that

$$\begin{aligned}\mathbf{s}_k(\mathbf{f})(x) = \sum_{n=0}^{k} \sum_{l=0}^{n} \Big(&\boldsymbol{\mathcal{S}}_{n,l}(x) \frac{1}{2\|\boldsymbol{\mathcal{S}}_{n,l}\|^2_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})}} \int_{\mathcal{S}} \overline{\mathbf{f}(y)} \boldsymbol{\mathcal{S}}_{n,l}(y) dV(y) \\ &+ \boldsymbol{\mathcal{S}}_{n,l}(x) \frac{1}{2\|\boldsymbol{\mathcal{S}}_{n,l}\|^2_{L^2(\mathcal{S};\mathbb{H};\mathbb{H})}} \int_{\mathcal{S}} \overline{\boldsymbol{\mathcal{S}}_{n,l}}(y) \mathbf{f}(y) dV(y)\end{aligned}$$

and, consequently, it holds

$$\begin{aligned}\mathbf{s}_k(\mathbf{f})(x) &= \sum_{n=0}^{k} \sum_{l=0}^{n} \boldsymbol{\mathcal{S}}^{n,l}(\mathbf{f}) + \int_{\mathcal{S}} \overline{\mathbf{G}_k(y)} \mathbf{f}(y) dV(y) \\ &= \mathbf{P}_k(\mathbf{f}) + \int_{\mathcal{S}} \overline{\mathbf{G}_k(y)} \mathbf{f}(y) dV(y) \;\longrightarrow\; \mathbf{P}(\mathbf{f}) + \mathbf{G}(\mathbf{f})\end{aligned}$$

in the L^2 sense as n approaches infinity. □

4.2. *Convergence aspects of prolate spheroidal monogenics*

This subsection begins with the fundamental definition of regular point of an $\mathbb{H}$-valued function.

Definition 4.3 (Regular Point). *Let $\mathbf{f}$ be an $\mathbb{H}$-valued function defined in $\mathcal{S}$. The point $y \in \mathcal{S}$ is called regular point for the function $\mathbf{f}$ if there exist the limits $\lim_{x \longrightarrow y+} \mathbf{f}(x) = \mathbf{f}(y+0)$ and $\lim_{x \longrightarrow y-} \mathbf{f}(x) = \mathbf{f}(y-0)$ (taken over all directions) such that $\mathbf{f}(y) = \frac{1}{2}\Big[\mathbf{f}(y+0) + \mathbf{f}(y-0)\Big]$.*

In the sequel we derive approximation properties for monogenic functions over 3D prolate spheroids, based on the Fourier expansion (4.3). Let $\mathcal{S}_{x,\delta}$ denote a neighbourhood of the point $x \in \mathcal{S}$ with diameter δ. The following assertions are valid:

Theorem 4.2. *Let $\mathbf{f} \in \mathcal{M}^+(\mathcal{S}; \mathbb{H})$ and $x \in \mathcal{S}$ a regular point of $\mathbf{f}$. For every $\delta > 0$ for which $\mathcal{S}_{x,\delta} \subseteq \mathcal{S}$, the integral*

$$\int_{\mathcal{S}_{x,\delta}} \frac{\mathbf{f}(x+y) + \mathbf{f}(x-y) - 2\mathbf{f}(x)}{|y|} \, dV(y)$$

is convergent. Then $S(\mathbf{f})(x)$ is convergent and $S(\mathbf{f})(x) \equiv \mathbf{f}(x)$.

Proof. From the definition of regular point it follows that

$$|\mathbf{f}(x+y) + \mathbf{f}(x-y) - 2\mathbf{f}(x)|$$

is enough closed to zero when $y \in \mathcal{S}_{x,\delta}$, and since $\left| \int_{-\epsilon}^{\epsilon} \frac{dy_j}{y_i} \right| = 0$ $(0 \leq i < j \leq 2)$ for every $\epsilon > 0$ enough closed to zero, the integrals

$$\int_{\mathcal{S}_{x,\delta}} \frac{\mathbf{f}(x+y) - \mathbf{f}(x)}{|y|} \, dV(y) \tag{4.6}$$

and

$$\int_{\mathcal{S}_{x,\delta}} \frac{\mathbf{f}(x-y) - \mathbf{f}(x)}{|y|} \, dV(y) \tag{4.7}$$

are convergent. Since (4.6) is convergent, there exists a $M_1 > 0$ so that for every $y \in \mathcal{S}_{x,\delta}$ it holds

$$|\mathbf{f}(x+y) - \mathbf{f}(x)| \leq M_1 |y|. \tag{4.8}$$

If for every $M > 0$ there exists $z \in \mathcal{S}_{x,\delta}$ such that $|\mathbf{f}(x+z) - \mathbf{f}(x)| > M|z|$, then the integral (4.6) is not convergent. Likewise, there exists a constant $M_2 > 0$ so that for every $y \in \mathcal{S}_{x,\delta}$ we have

$$|\mathbf{f}(x-y) - \mathbf{f}(x)| \leq M_2 |y|. \tag{4.9}$$

Consequently, from (4.8) and (4.9) it follows that $\mathbf{f}$ satisfies the Hölder's condition of exponent 1. Moreover, since $S(\mathbf{f})(x)$ is the expansion of $\mathbf{f}$ using the orthogonal basis (4.2) and by definition of regular point (in all directions), it follows that $S(\mathbf{f})(x)$ is convergent. In addition, since $\mathbf{f}$ satisfies the Hölder's condition, it holds $S(\mathbf{f})(x) \equiv \mathbf{f}(x)$. □

The result stated above suggests in a natural way the following.

Theorem 4.3. *Let $\mathbf{f} \in \mathcal{M}^+(\mathcal{S}; \mathbb{H})$ and $x \in \mathcal{S}$ a regular point of $\mathbf{f}$. If there exists a $\delta > 0$ for which $\mathcal{S}_{x,\delta} \subseteq \mathcal{S}$ and $\mathbf{f}(x) \equiv \boldsymbol{0}_{\mathbb{H}}$ on $\mathcal{S}_{x,\delta}$ then $S(\mathbf{f})(x)$ is convergent and $S(\mathbf{f})(x) \equiv \boldsymbol{0}_{\mathbb{H}}$.*

Proof. The proof follows from Theorem 4.2. □

Corollary 4.1. *Let* $\mathbf{f}, \mathbf{g} \in \mathcal{M}^+(\mathcal{S}; \mathbb{H})$. *If there exists a point* $x \in \mathcal{S}$ *such that there exists a* $\delta > 0$ *so that* $\mathcal{S}_{x,\delta} \subseteq \mathcal{S}$ *and* $\mathbf{f} \equiv \mathbf{g}$ *on* $\mathcal{S}_{x,\delta}$, *then* $S(\mathbf{f})(x)$ *and* $S(\mathbf{g})(x)$ *are convergent or* $S(\mathbf{f})(x) \equiv S(\mathbf{g})(x) = \pm\infty$.

Proof. The proof follows from the previous theorem for $\mathbf{f} - \mathbf{g}$. □

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