Uncertainty principles for hypercomplex signals in the linear canonical transform domains

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Abstract

Linear canonical transforms (LCTs) are a family of integral transforms with wide application in optical, acoustical, electromagnetic, and other wave propagation problems. The Fourier and fractional Fourier transforms are special cases of LCTs. In this paper, we extend the uncertainty principle for hypercomplex signals in the linear canonical transform domains, giving the tighter lower bound on the product of the effective widths of complex paravector- (multivector-)valued signals in the time and frequency domains. It is seen that this lower bound can be achieved by a Gaussian signal. An example is given to verify the result.

1. Introduction

Uncertainty principle in the time–frequency plane plays an important role in signal processing [14,17,22,23,32,38,44,47,54,55,60]. This principle states that for a given unit energy signal \( f(t) \) with Fourier transform

\[
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt,
\]

the product of spreads of the signal in time domain and frequency domain is bounded by a lower bound:

\[
\sigma_t^2 \sigma_\omega^2 \geq \frac{1}{4}
\]

(1)

where \( \sigma_t \) and \( \sigma_\omega \) are the duration and bandwidth of a signal \( f(t) \) defined by

\[
\sigma_t^2 = \int_{-\infty}^{\infty} (t - \langle t \rangle)^2 |f(t)|^2 dt
\]

\[
\sigma_\omega^2 = \int_{-\infty}^{\infty} (\omega - \langle \omega \rangle)^2 |\hat{f}(\omega)|^2 d\omega
\]

respectively. Here

\[
\langle t \rangle = \int_{-\infty}^{\infty} t |f(t)|^2 dt
\]

is the mean time and

\[
\langle \omega \rangle = \int_{-\infty}^{\infty} \omega |\hat{f}(\omega)|^2 d\omega
\]

is the mean frequency. Without loss of generality, let \( \langle t \rangle = 0 \) and \( \langle \omega \rangle = 0 \), then the essence of uncertainty principle (1) will not be affected. The reason is that the standard derivation does not depend on the mean because it is defined as the broadness about the mean [10,18]. Consequently, Eq. (1) becomes

\[
\left( \int_{-\infty}^{\infty} t^2 |f(t)|^2 dt \right) \left( \int_{-\infty}^{\infty} \omega^2 |\hat{f}(\omega)|^2 d\omega \right) \geq \frac{1}{4}
\]

For the importance of uncertainty principle in physics

[1,10,28–30,39,40,51,56,59], there are many efforts to extend it to various types of functions and integral transformations.
The linear canonical transform (LCT) being a generalization of the Fourier transform (FT), the fractional Fourier transform (FrFT) and the Fresnel transform [48,44], was first proposed in the 1970s by Moshinsky and Collins [13,43]. It is an effective processing tool for chirp signal analysis, such as the LCT filtering [21,49,58], the parameter estimation and sampling progress for non-bandlimited signals with nonlinear Fourier atoms [37]. The windowed LCT [35], with a local window function, can reveal the local LCT-frequency contents, and it enjoys high concentrations and eliminates cross terms. The analogue of the Poisson summation formula, sampling formulas, series expansions and Paley–Wiener theorem is studied in [35,36]. As a powerful analyzing tool, LCT has been widely used in many fields such as optics and signal processing [3,43,50,57]. Recently, researchers discussed the uncertainty relations for FrFT [44,49,54] and LCT [36,55,53,56] and tried to derive sharp uncertainty principle bounds for them. A stronger uncertainty principle in LCT involving the phase derivative of the signal was discussed in [15].

In view of numerous applications, one is particularly interested in higher dimensional analogues to Euclidean space $\mathbb{R}^n$. Higher dimensional extensions of the LCT have been studied, for instance, in [33,34]. The LCT is first extended to the Clifford algebra $Cl_{0,m}$ (see the notation in Section 2) in [34]. It is used to study the generalized prolate spheroidal wave functions and the connection with energy concentration problems [34]. The present work develops the definition of LCT (see Definition 3.1 in Section 3) which uses the imbedding of $\mathbb{R}^n$ into the (complex) Clifford algebra $Cl_{0,m}$. The Clifford algebra $Cl_{0,m}$ provides $\mathbb{R}^n$ with a global complex structure in analogy with the imbedding of $\mathbb{R}$ into the complex plane $\mathbb{C}$. Under this frame we present in this note the precise analogue of the classical Heisenberg uncertainty principles in linear canonical domains which have been targeted by others.

In Hamiltonian quaternion analysis some papers combined the uncertainty relations and the quaternion Fourier transforms (QFTs) [2,7,25,46]. Due to the non-commutative property of multiplication of quaternions, there are different types of QFTs: double-sided (two-sided) QT, left-sided QT and right-sided QT. The QT plays a vital role in the representation of hypercomplex signals. It transforms a real (or quaternionic) 2D signal into a quaternion-valued frequency domain signal. The four components of the QT separate four cases of symmetry into real signals instead of only two as in the complex Fourier transform. In [52] the authors used the QT to proceed color image analysis. The paper [6] implemented the QT to design a color image digital watermarking scheme. The QT are applied to image pre-processing and neural computing techniques for speech recognition [5]. It is well-known that the Plancherel theorem is not valid for the double-sided or the left-sided QT [2]. For this reason, many studies focus on the right-sided QT. In [19,20], certain asymptotic properties of the (right-sided) QT are analyzed and generalizations of classical Bochner–Minlos theorems to the framework of quaternionic analysis are derived. The uncertainty principle for the (right-sided) QLCT, the generalization of the (right-sided) QT in the Hamiltonian quaternion algebra, is recently derived in [33]. In 2010, Hitzer [25] studied the $\pm$ split of quaternions and its effects on the double-sided QFT. Based on this he formulated the directional uncertainty principle for QT of quaternion-valued 2D signal. The QT belongs to the growing family of Clifford Fourier transformations.

In Clifford algebra, Hitzer et al. [26,27,41,42] investigated a directional uncertainty principle for the Clifford Fourier transform, which describes how the variances (in arbitrary but fixed directions) of a multivector-valued function and its Clifford Fourier transform are related. In [26,27], the research was studying for the Clifford- ($Cl_{0,m}$, $m = 2, 3$ (mod 4))-valued signals, while the latter [41,42] focused on the octonion-($Cl_{0,3}$)-valued signals. Using the scalar-valued phase derivative of hypercomplex signals [62], two uncertainty principles, of which one is for scalar-valued hypercomplex signals and the other is for axial form hypercomplex signals, for Fourier transforms were studied in [61]. To the best of our knowledge, a systematic work on the investigation of uncertainty relations using the LCT of a paravector- (multivector-)valued function has not been carried out yet.

In the present work, we study the LCT in Euclidean space which transforms a paravector-valued space signal into a complex paravector-valued frequency signal. Some important properties of the LCT are analyzed. Two uncertainty principles for the LCT of complex multivector-valued signals are established. These uncertainty principles prescribe the lower bounds on the products of the effective widths of complex multivector-valued signals in the space and frequency domains. The main motivation of the present study is to develop further general methods for time–frequency analysis, developing sampling theory in the m-D case, filter design, signal synthesis and optics in the Clifford algebra. We note that the present theory for the paravector- (multivector-)valued signals can also be generalized to the complex Clifford- ($Cl_{0,m}$)-valued signals without difficulties.

The article is organized as follows. In Section 2 we provide the basic knowledge of Clifford algebra used in the paper. Then the LCT of complex paravector-valued signal is introduced and studied in Section 3. Some important properties such as Parseval theorem are obtained. They are necessary to prove the uncertainty principle in the LCT domain. In Section 4 we formulate and prove the classical Heisenberg uncertainty principles for the LCT of complex paravector-valued signal. These principles prescribe lower bounds on the products of the effective widths of paravector-valued signals in the time and frequency domains. We give an example to verify the result in Section 5. Some conclusions are drawn in Section 6.

2. Clifford algebra

The theory of Clifford algebras is intimately connected with the theory of quadratic forms and orthogonal transformations. They generalize the real numbers, complex numbers, quaternions and several other hypercomplex number systems [11,12]. Clifford algebras have important applications in a variety of fields including geometry.
theoretical physics and digital image processing. They are named after the English geometer William Kingdon Clifford [11,12]. In the present section, we begin by reviewing some definitions and basic properties of Clifford algebra [9,16].

For all what follows we will work in $\mathbb{R}^m (C^m)$, the usual Euclidean space (several complex variables space). This means we can express each element $\bar{x} \in \mathbb{R}^m (C^m)$ uniquely in the form:

$$\bar{x} = x_0 e_0 + \sum_{j=1}^m x_j e_j, \quad x_j \in \mathbb{R} (C) \quad (j = 1, 2, \ldots, m)$$

where the units $e_1, \ldots, e_m$ are basic elements of $\mathbb{R}^m (C^m)$ satisfying

$$e_i^2 = -1 \quad \text{and} \quad e_i e_j = -e_j e_i, \quad i \neq j, \quad (i, j = 1, 2, \ldots, m).$$

The real (complex) Clifford algebra generated by $e_1, e_2, \ldots, e_m$, denoted by $Cl_0, m$, is the associative algebra over the real (complex) field $\mathbb{R} (C)$. Clearly, it is non-commutative. A general element in $Cl_0, m$, therefore, is of the form $x = \sum_{j=1}^m x_j e_j, \quad x_j \in \mathbb{R} (C)$ and $e_0 = e_1 e_2 \ldots e_m$, and $S$ runs over all the ordered subsets of $\{1, 2, \ldots, m\}$, namely $S = \{\bar{S} \in \{1, 2, \ldots, i_2 \ldots i_m\} \quad 1 \leq i_1 < \ldots < i_m \leq m\} \quad 1 \leq i_m \leq m$. The conjugation of $e_0$ is defined by $e_0^* = -e_0$. So the Clifford conjugate of a vector $\bar{x} \in \mathbb{R}^m$ is $\bar{x} = -\bar{x}$, while the Clifford conjugate of a complex vector $\bar{x} \in C^m$ is $\bar{x} = -\bar{x} e_0$.

The real (complex) paravector space $\mathbb{R}^m (C^m + 1)$ is the linear subspace defined by

$$\mathbb{R}^m (C^m + 1) = \text{span} \{e_0 (1, e_1, \ldots, e_m) \in Cl_0, m\} \quad \text{with} \quad e_0 (1, e_1, \ldots, e_m) \quad \text{as} \quad \text{elements of the form} \quad x = x_0 + x, \quad x = x_0 + x \in \mathbb{R} (C) \quad \text{and} \quad x \in \mathbb{R}^m (C^m).$$

The scalar and vector parts of $x, \text{Sc} (x)$ and $\text{Vec} (x)$, are defined as the $x_0$ and $\bar{x}$ terms, respectively. We shall always assume the paravector $0 + 0e_0 + \cdots + 0e_m = 0$ to be the neutral element of addition in the sequel.

The multiplication of two complex paravectors $x = x_0 + x, \quad y = y_0 + y \in \mathbb{R}^m (C^m + 1)$ is given by

$$xy = (x_0 + x)(y_0 + y) = x_0 y_0 + x_0 y + y_0 x + x y$$

In particular, for $y = \bar{x} = x_0 + \bar{x} \in \mathbb{R}^m (C^m + 1)$, we have

$$x x \bar{x} = x_0 x_0 x_0 + x_0 \bar{x} + \bar{x} + x \bar{x}$$

$$= (x_0 x_0 + \bar{x}) + (x_0 + \bar{x}) = x_0 + x \bar{x} + x_0 + \bar{x} \bar{x} \quad \text{and} \quad x / x = \sum_{j=0}^m x_0 x_j e_j$$

where $x \cdot \bar{x} = x_1 x_1 + x_2 x_2 \ldots \ldots + x_m x_m$ and $x_0 x \bar{x} + \bar{x} + x \bar{x} + \bar{x}$.

Unlike the case in the usual complex space, the norm $|x|$ of $x \in \mathbb{R}^m (C^m + 1)$ is defined by

$$|x| = \sqrt{\text{Sc}(x \bar{x})} = \sqrt{\text{Sc}(x \bar{x})} = \sqrt{|x_0|^2 + |x_1|^2 + \cdots + |x_m|^2}.$$
it is natural to extend it to higher dimensional spaces. Higher dimensional extensions of the LCT has been studied in [33,34]. We use the special case of the LCT in [34].

**Definition 3.1.** Let 
\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2 \times 2} \]
be a matrix parameter such that \( \det(A) = 1 \). The LCTs of hypercomplex signals \( f \in L^1(\mathbb{R}^m; \mathbb{C}^{m+1}) \) are defined by
\[
L(f)(y) = \frac{1}{\sqrt{(2\pi)^m b}} \int_{\mathbb{R}^m} f(x) e^{i k_a (x_b y - 1 / (2b) x_c)} dx,
\]
and, if \( L(f) \in L^1(\mathbb{R}^m; \mathbb{C}^{m+1}) \), the inverse LCTs are defined by
\[
L^{-1}(f)(y) = \frac{1}{\sqrt{(2\pi)^m b}} \int_{\mathbb{R}^m} f(x) e^{-i k_a (x_b y - 1 / (2b) x_c)} dx.
\]

Note that for \( b = 0 \) the LCT of a signal is essentially a chirp multiplication and it is of no particular interest for our objective in this work. Hence, without loss of generality, we set \( b \neq 0 \) in the following sections unless stated. Therefore
\[
L(f)(y) = \int_{\mathbb{R}^m} f(x) K(x, y) dx
\]
with the kernel function
\[
K(x, y) := \frac{1}{\sqrt{(2\pi)^m b}} e^{i k_a (x_b y - 1 / (2b) x_c) + (d / 2b) x_d} dx, b \neq 0.
\]

It is significant to note that when \( a = d = 0, b = 1 \), the LCT of \( f \) reduces to the FT of \( f \) in \( \mathbb{R}^m \). That is
\[
L(f)(y) = \frac{1}{\sqrt{(2\pi)^m b}} \int_{\mathbb{R}^m} f(x) e^{-i k_a (x_b y)} dx = \frac{1}{\sqrt{b}} F\{ f \}(y).
\]
where \( F\{ f \} \) is the FT of \( f \) given by (3).

Furthermore, they have the following close relationship.

**Lemma 3.1.** Let
\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2 \times 2} \]
be a matrix parameter such that \( \det(A) = 1 \) and \( b \neq 0 \). Let \( f \in L^1(\mathbb{R}^m; \mathbb{C}^{m+1}) \), then
\[
L(f)(y) = \frac{1}{\sqrt{(2\pi)^m b}} \int_{\mathbb{R}^m} f(x) e^{i k_a (x_b y)} dx.
\]

**Proof.** By the definition of \( L(f) \) in (5), a direct computation shows that
\[
L(f)(y) = \int_{\mathbb{R}^m} f(x) K(x, y) dx
\]
\[
= \frac{1}{\sqrt{(2\pi)^m b}} \int_{\mathbb{R}^m} f(x) e^{i k_a (x_b y)} e^{-i x_c / (2b)} dx
\]
\[
= \frac{1}{\sqrt{(2\pi)^m b}} \int_{\mathbb{R}^m} f(x) e^{i k_a (x_b y)} e^{-i x_c / (2b)} dx
\]
\[
= \frac{1}{\sqrt{(2\pi)^m b}} \int_{\mathbb{R}^m} f(x) e^{i k_a (x_b y)} e^{-i x_c / (2b)} dx
\]
Therefore the result (7) follows. □

**Remark 3.1.** Since the classical FT (3) is the special case of LCT (5) and the (real) Clifford FT [26] has an isomorphism with the FT (3), the present definition of LCT (5) generalizes the (real) Clifford FT [26].

We then establish the Plancherel theorem for LCT of hypercomplex signals.

**Theorem 3.1 (Plancherel theorem).** If \( f, f \in L^2(\mathbb{R}^m; \mathbb{C}^{m+1}) \), then
\[
\langle f, f \rangle_{L^2(\mathbb{R}^m; \mathbb{C}^{m+1})} = \langle L(f), L(f) \rangle_{L^2(\mathbb{R}^m; \mathbb{C}^{m+1})}.
\]

Particularly, if \( f = f \), then the Parseval theorem is obtained. That is
\[
\| f \|_{L^2(\mathbb{R}^m; \mathbb{C}^{m+1})} = \| L(f) \|_{L^2(\mathbb{R}^m; \mathbb{C}^{m+1})}.
\]

**Proof.** By the inner product (2) and (7) in Lemma 3.1, we have
\[
\langle L(f), L(f) \rangle_{L^2(\mathbb{R}^m; \mathbb{C}^{m+1})} = \int_{\mathbb{R}^m} \mathrm{Sc} \left( \frac{1}{\sqrt{b}} F\{ f \} \right) \left( \frac{1}{\sqrt{b}} F\{ f \} \right) \left( \frac{1}{\sqrt{b}} \right) \left( \frac{1}{\sqrt{b}} \right) dx
\]
Since the complex value \((1 / \sqrt{b}) F\{ f \} \) computes with any paravector-valued signals, the above becomes
\[
\langle L(f), L(f) \rangle_{L^2(\mathbb{R}^m; \mathbb{C}^{m+1})} = \int_{\mathbb{R}^m} \mathrm{Sc} \left( \frac{1}{\sqrt{b}} F\{ f \} \right) \left( \frac{1}{\sqrt{b}} F\{ f \} \right) \left( \frac{1}{\sqrt{b}} \right) \left( \frac{1}{\sqrt{b}} \right) dx
\]
where we have used the Plancherel theorem of Fourier transforms of \( f \) in \( L^2(\mathbb{R}^m; \mathbb{C}^{m+1}) \). □

Theorem 3.1 shows that the total signal energy computed in the time domain equals to the total signal energy in the frequency domain, and the change of domains for convenience of computation.

To proceed with, we prove the following partial derivative properties.

**Lemma 3.2.** For \( k = 1, \ldots, m \), if \( f \) and \( \partial / \partial x_k \in L^1(\mathbb{R}^m; \mathbb{C}^{m+1}) \), then
\[
F\left( \frac{\partial}{\partial x_k} (f e^{i k_a (x_b y)}) \right) (y) = \mathrm{ii}_k F\{ f \} (y).
\]

**Proof.** Applying the integration by parts and complex value \(-\mathrm{ii}_k\) computes with any paravector-valued signals, we obtain
\[
F\left( \frac{\partial}{\partial x_k} (f e^{i k_a (x_b y)}) \right) (y) = \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} \frac{\partial}{\partial x_k} \left( f e^{i k_a (x_b y)} \right) e^{-i k_a x} dx
\]
\[
= -\frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} \left( -\mathrm{ii}_k f \right) e^{i k_a (x_b y)} e^{-i k_a x} dx
\]
Applying (8) in Lemma 3.2 and Parseval theorem of Fourier

Then

For any real number \( r \), we have

\[
\int_{R^n} |L(f)(u)|^2 \, du = b^2 \int_{R^n} \left| \left( \frac{\partial}{\partial \alpha_0} f(\chi)e^{ik_0/2b^2} \right) \right|^2 \, d\alpha_0.
\]

Lemma 3.3. For \( k = 1, \ldots, m \), suppose \( f \in L^1(R^n; C^{m+1}) \), \( u_k L(f) \) and \( \frac{\partial}{\partial \alpha_k} \in L^2(R^n; C^{m+1}) \), then

\[
\int_{R^n} u_k^2 |L(f)(u)|^2 \, du = b^2 \int_{R^n} \left| \frac{\partial}{\partial \alpha_k} f(\chi) - \frac{\partial}{\partial \alpha_k} \chi_0 f(\chi) \right|^2 \, d\chi.
\]

Proof. By (7) in Lemma 3.1, we have

\[
\int_{R^n} u_k^2 |L(f)(u)|^2 \, du = \int_{R^n} u_k^2 \left| \frac{1}{\sqrt{b^2}} e^{ik/2b^2} F \left\{ \left( \frac{\partial}{\partial \alpha_0} f(\chi)e^{ik_0/2b^2} \right) \right\} \right|^2 \, du
\]

\[
= b^2 \int_{R^n} \left| \left( \frac{\partial}{\partial \alpha_0} f(\chi)e^{ik_0/2b^2} \right) \right|^2 \, d\alpha_0.
\]

\[
= b^2 \int_{R^n} |\partial f(\chi)e^{ik_0/2b^2} | \, d\alpha_0.
\]

\[
= b^2 \int_{R^n} |\partial f(\chi) - \frac{\partial}{\partial \alpha_k} \chi_0 f(\chi) |^2 \, d\chi.
\]

Applying (8) in Lemma 3.2 and Parseval theorem of Fourier transform of \( \frac{\partial}{\partial \alpha_0} f(\chi)e^{ik_0/2b^2} \), we obtain

\[
\int_{R^n} u_k^2 |L(f)(u)|^2 \, du = b^2 \int_{R^n} \left| \left( \frac{\partial}{\partial \alpha_0} f(\chi)e^{ik_0/2b^2} \right) \right|^2 \, d\alpha_0
\]

We are now in the heart of the matter.

4. Uncertainty principles

In signal processing much effort has been placed in the study of the classical Heisenberg uncertainty principle during the last years. To our knowledge, a systematic work on the investigation of uncertainty relations using the LCT of hypercomplex signal is not carried out.

In the following we explicitly prove and generalize the classical uncertainty principle to Clifford module functions using the LCTs. We also give an explicit proof for Gaussian signals (Gabor filters) to be indeed the only signals that minimize the uncertainty.

First, we assume signals that have zero mean spaces:

\[
0 = \langle \chi_k \rangle = \int_{R^n} \chi_k f(\chi) \, d\chi
\]

and

\[
0 = \langle \chi \rangle = \int_{R^n} \chi f(\chi) \, d\chi.
\]

Furthermore, zero mean frequencies:

\[
0 = \langle u_k \rangle = \int_{R^n} u_k L(f)(u) \, du
\]

and

\[
0 = \langle u \rangle = \int_{R^n} u L(f)(u) \, du.
\]

Let us now begin the proofs of two uncertainty relations.

Theorem 4.1. For \( k = 1, \ldots, m \), suppose unit energy signal \( f \in L^1(R^n; C^{m+1}) \), \( u_k L(f) \), \( \chi_0 f \) and \( \partial / \partial \alpha_k \in L^2(R^n; C^{m+1}) \), then

\[
\left( \int_{R^n} |x_k^2 f(\chi)|^2 \, d\chi \right) \left( \int_{R^n} u_k^2 |L(f)(u)|^2 \, du \right) \geq b^2/4.
\]

and equality is achieved when

\[
f(\chi) = C_0 \exp \left( -1/2 (\alpha_1 x_1^2 + \ldots + \alpha_m x_m^2) \right),
\]

where \( \alpha_k \) are positive real constants and \( C_0 = (\alpha_1 \cdots \alpha_m / \pi^2)^{1/4} \).

Proof. Applying (9) in Lemma 3.3, and Schwarz’s inequality (10), we have

\[
\left( \int_{R^n} |x_k^2 f(\chi)|^2 \, d\chi \right) \left( \int_{R^n} u_k^2 |L(f)(u)|^2 \, du \right) = b^2 \left( \int_{R^n} x_k f(\chi) \, d\chi \right) \left( \int_{R^n} \left| \frac{\partial}{\partial \alpha_k} \chi_0 f(\chi) \right|^2 \, d\chi \right)
\]
\[ \begin{align*}
&\geq \frac{b^2}{4} \left( \int_{R^n} \text{Sc} \left( x_nf(x) \frac{\partial}{\partial x_k} f(x) - \frac{\partial}{\partial x_k} x_nf(x) \right) \right) \\
&\quad + \left[ \frac{\partial}{\partial x_k} f(x) - i \frac{\partial}{\partial x_k} x_nf(x) \right] x_nf(x) \int_{R^n} dx^2 . \end{align*} \] (17)

By direct calculation,
\[ \begin{align*}
\left| \int_{R^n} \text{Sc} \left( x_nf(x) \frac{\partial}{\partial x_k} f(x) - i \frac{\partial}{\partial x_k} x_nf(x) \right) \right| \\
&= \int_{R^n} \text{Sc} \left( \left[ f(x) \frac{\partial}{\partial x_k} f(x) - i \frac{\partial}{\partial x_k} x_nf(x) \right] x_nf(x) \right) \\
&= \int_{R^n} \text{Sc} \left( \left[ f(x) \frac{\partial}{\partial x_k} f(x) - i \frac{\partial}{\partial x_k} x_nf(x) \right] \right) \\
&= 1 . \end{align*} \] (18)

The first term of (18) is a perfect differential and integrates to zero. The second term gives one half since we assume the signal is unit energy. Therefore we have the result (15) follows.

Since the minimum value for the uncertainty product is \( b^2/4 \), we can ask what signals have that minimum value. The Schwarz’s inequality (17) becomes an equality when the two functions are proportional to each other. Hence we take

\[ -C_x f(x) = \frac{\partial}{\partial x_k} f(x) - i \frac{\partial}{\partial x_k} x_nf(x) . \]

where \( C \) is an arbitrary constant. Solving this differential equation, we get
\[ f(x) = C_0 \exp \left( \frac{-1}{2} \left( C_1 - i \frac{\alpha}{\beta} \right) x_1^2 + \cdots + \left( C_n - i \frac{\alpha}{\beta} \right) x_n^2 \right) . \]

where \( C_0 \) is a constant of integration. In order to make \( f \in L^1(R^n, C^{m+1}) \), \( C_k \) can be chosen such that
\[ \alpha_k = C_k - i \frac{\alpha}{\beta} . \]

where \( \alpha_k \) is positive real constant.
\[ f(x) = C_0 \exp \left( -i \frac{\alpha}{\beta} x_1^2 + \cdots + \alpha n x_n^2 \right) . \] (19)

The value of \( C_0 \) can be found out by noting that \( f \) must be unit norm. That is, we have
\[ C_0 \equiv \left( \frac{\alpha_1 \cdots \alpha_m}{\pi^2} \right)^{1/4} . \]

Thus, \( f(x) \) given in (19), which satisfies the Cauchy–Schwarz inequality (17), turns out to be a Gaussian function and the theorem is proved. \( \square \)

Since the Gaussian function \( f(x) \) of (19) achieves the minimum width-bandwidth product, it is theoretically a very good prototype wave form. One can therefore construct a basic wave form using spatially or frequency scaled versions of \( f(x) \) to provide multiscale spectral resolution. In quaternion analysis, such a wavelet basis construction derived from a Gaussian function prototype wave form has been realized, for example, in the quaternionic wavelet transforms in [4]. The optimal space-frequency localization is also another reason why 2D Clifford–Gabor bandpass filters were suggested in [8].

We now derive a new directional uncertainty principle for complex paravector-valued signals subject to the linear canonical transformation.

**Theorem 4.2.** For \( k = 1, \ldots, m \), suppose \( f \in L^1(R^m, C^{m+1}) \), \( u_i L(f), x_i f \) and \( df/dx_k \in L^2(R^m, C^{m+1}) \), then
\[ \left( \int_{R^m} |x|^2 |f(x)|^2 dx \right)^{1/2} \left( \int_{R^m} |u|^2 |L(f)(u)|^2 du \right)^{1/2} \geq \frac{b^2 m^2}{4} . \] (20)

and equality is achieved when \( f \) is a Gaussian function, i.e.,
\[ f(x) = C_0 \exp \left( -i \frac{1}{2} (\alpha_1 x_1^2 + \cdots + \alpha_m x_m^2) \right) . \] (21)

where \( \alpha_k \) are positive real constants and \( C_0 = (\alpha_1 \cdots \alpha_m / \pi^2)^{1/4} \).

**Proof.** Applying (9) in Lemma 3.3, we have
\[ \left( \int_{R^m} |x|^2 |f(x)|^2 dx \right)^{1/2} \left( \int_{R^m} |u|^2 |L(f)(u)|^2 du \right)^{1/2} \]
\[ \quad = \left( \int_{R^m} \sum_{k=1}^m |x_k|^2 |f(x_k)|^2 dx \right)^{1/2} \left( \int_{R^m} \sum_{k=1}^m |u_k|^2 |L(f)(u_k)|^2 du \right)^{1/2} \]
\[ \quad = b^2 \left( \int_{R^m} \sum_{k=1}^m |x_k|^2 |f(x_k)|^2 \right)^{1/2} \left( \int_{R^m} \sum_{k=1}^m \left| \frac{\partial}{\partial x_k} f(x) - i \frac{\partial}{\partial x_k} x_k f(x) \right|^2 \right)^{1/2} \]
\[ \quad \geq b^2 \left( \int_{R^m} \sum_{k=1}^m |x_k|^2 |f(x_k)|^2 \right)^{1/2} \left( \int_{R^m} \sum_{k=1}^m \left| \frac{\partial}{\partial x_k} f(x) - i \frac{\partial}{\partial x_k} x_k f(x) \right|^2 \right)^{1/2} \]
\[ \quad \geq b^2 . \] (22)

where we have used the Schwarz inequality of square integrable real valued signals. For \( i = 1, \ldots, m \), the Schwarz inequality of paravector-numbers \( s_i \) and \( t_i \) is given by
\[ \left( \frac{1}{2} \sum_{i=1}^m s_i t_i + i \sum_{i=1}^m s_i \bar{t}_i \right)^2 \leq \left( \sum_{i=1}^m |s_i|^2 \right) \left( \sum_{i=1}^m |t_i|^2 \right) . \] (23)

Equality holds if and only if \( s_k \) and \( t_k \) are linearly dependent. One has
\[ \left( \sum_{k=1}^m \left| x_k^2 f(x_k) \right|^2 \right)^{1/2} \left( \sum_{k=1}^m \left| \frac{\partial}{\partial x_k} f(x) - i \frac{\partial}{\partial x_k} x_k f(x) \right|^2 \right)^{1/2} \]
\[ \geq \frac{1}{2} \sum_{k=1}^m \text{Sc} \left( x_k f(x) \frac{\partial}{\partial x_k} f(x) - i \frac{\partial}{\partial x_k} x_k f(x) \right) \]
\[ + \left( \frac{\partial}{\partial x_k} f(x) - i \frac{\partial}{\partial x_k} x_k f(x) \right) \left( x_k f(x) \right) \]
\[ = \frac{1}{2} \sum_{k=1}^m \text{Sc} \left( x_k f(x) \frac{\partial}{\partial x_k} f(x) + \frac{\partial}{\partial x_k} f(x) x_k f(x) \right) \]
\[ = \frac{1}{2} \sum_{k=1}^m x_k f(x) \frac{\partial}{\partial x_k} f(x) - f(x) \frac{\partial}{\partial x_k} f(x) \right) . \] (24)

Therefore (22) becomes
\[ \left( \int_{R^m} |x|^2 |f(x)|^2 dx \right)^{1/2} \left( \int_{R^m} |u|^2 |L(f)(u)|^2 du \right)^{1/2} \geq \frac{b^2 m^2}{4} . \]

We finally show that equality in (22) and (24) are satisfied if and only if \( f \) is a Gaussian function given by (21).

Schwarz’s inequality (10) of real valued signals becomes an equality when the two functions are proportional to each other. Hence we have
\[ -C_x f(x) = \frac{\partial}{\partial x_k} f(x) - i \frac{\partial}{\partial x_k} x_k f(x) , \]
where $C_k$ is an arbitrary constant. Solving this differential equation, we get

$$f(x) = C_0 \exp \left( \frac{1}{2} \sum a_k x_k^2 \right),$$

where $a_k$ are positive real constant and

$$C_0 = (\alpha_1 \cdots \alpha_m)^{1/4}.$$

Thus, $f(x)$ given by (21), which satisfies the Cauchy–Schwarz inequality (17), turns out to be a Gaussian function and the theorem is proved. □

When $a = 0$, $b = 1$, we have

**Corollary 4.1.** For $k = 1, 2, \ldots, m$, if $f \in L^1(\mathbb{R}^m, \mathbb{C}^{m+1}), x_kf$ and $\langle f, \alpha_k \rangle \in L^2(\mathbb{R}^m, \mathbb{C}^{m+1})$, then

$$\left( \int_{\mathbb{R}^m} |x^2| |f(x)|^2 \, dx \right) \left( \int_{\mathbb{R}^m} |u^2| |\hat{f}(u)|^2 \, du \right) \geq \frac{1}{4} \alpha_k^2.$$

**Remark 4.1.**

(i) Note that these uncertainty relations (Theorems 4.1 and 4.2) are consistent for the complex-valued signals $(m = 1)$. While Theorem 4.2 formulates a directional uncertainty principle for complex paravector-valued signals and Theorem 4.1 gives the spatial case.

(ii) Note that many studies of uncertainty principles in the Clifford algebra, most of the researches cannot be reduced to the 1D $(m = 1)$ case, for instance [26,27,41,42].

Furthermore, we prove that the standard deviation does not depend on the mean because it is defined as the broadness about the mean. We then obtain the general results for the uncertainty relations.

**Theorem 4.3.** For $k = 1, \ldots, m$, suppose unit energy signal $f \in L^1(\mathbb{R}^m, \mathbb{C}^{m+1}), u_kf$, $x_kf$ and $\langle f, \alpha_k \rangle \in L^2(\mathbb{R}^m, \mathbb{C}^{m+1})$, then we have

(i) \[ \left( \int_{\mathbb{R}^m} |x_k(x_k - x_k)|^2 |f(x)|^2 \, dx \right) \left( \int_{\mathbb{R}^m} |u_k(u_k - u_k)|^2 |\hat{f}(u)|^2 \, du \right) \geq \frac{b^2}{4}. \]

(ii) \[ \left( \int_{\mathbb{R}^m} |x_k(x_k)^2 |f(x)|^2 \, dx \right) \left( \int_{\mathbb{R}^m} |u_k(u_k)^2 |\hat{f}(u)|^2 \, du \right) \geq \frac{b^2 m^2}{4}. \]

Furthermore, equalities of (i) and (ii) are achieved when

$$f(x) = C_0 \exp \left( \frac{1}{2} \sum a_k x_k^2 \right),$$

where $\langle x_k \rangle, \langle u_k \rangle, \langle f \rangle$ are defined by (11), (12), (13) and (14), respectively, the constants $\alpha_k$ are positive real numbers and $C_0 = (\alpha_1 \cdots \alpha_m)^{1/4}$.

**Proof.** The proof of part (i) works very similar to the classical case in the complex plane. We therefore do not repeat it here.

To prove (ii), if we take a signal $f$ that has zero mean space (12) and zero mean frequency (14), we want a signal of the same shape but with particular mean space $\langle x \rangle$ and frequency $\langle u \rangle$, then a new signal can be defined by

$$g(x) = e^{ik1(x_1 - \langle x \rangle_1)} e^{ik2(x_2 - \langle x \rangle_2)} f(x - \langle x \rangle).$$

Therefore

$$\int_{\mathbb{R}^m} |x - \langle x \rangle|^2 |g(x)|^2 \, dx = \int_{\mathbb{R}^m} |x_k(x_k - \langle x \rangle)|^2 |f(x - \langle x \rangle)|^2 \, dx$$

and

$$\int_{\mathbb{R}^m} |u_k(u_k - \langle u \rangle)|^2 |L(g)(u)|^2 \, du = \int_{\mathbb{R}^m} |u_k(u_k - \langle u \rangle)|^2 |\hat{f}(\langle u \rangle)|^2 \, du.$$

Here we use the property $|L(g)(u)| = |L(f)(u - \langle u \rangle)|$. In fact, by Eq. (5), we have

$$|L(g)(u)| = \left| \int_{\mathbb{R}^m} g(x) K(\langle x \rangle, u) \, dx \right|$$

and

$$\int_{\mathbb{R}^m} f(x - \langle x \rangle) K(\langle x \rangle, \langle u \rangle) \, dx = \int_{\mathbb{R}^m} f(x - \langle x \rangle - \langle u \rangle) \, dx.$$

\[ f_{\mathbb{R}^n} \frac{|x|^2}{|x| + x^{2m+2}} \, dx = \frac{s^2}{|x| + x^{2m+2}} \, dx \]

and

\[ \frac{1}{\|L\|_2^2} \int_{\mathbb{R}^n} |x|^2 |L[f](x)|^2 \, dx = \frac{1}{\|L\|_2^2} \int_{\mathbb{R}^n} |y|^2 |L[f](y)|^2 \, dy \]

\[ \frac{b^2 f_{\mathbb{R}^n} |z|^2 e^{-2\pi i z \cdot d} \, dz}{\|f\|_2^2} = \frac{b^2 f_{\mathbb{R}^n} |z|^2 e^{-2\pi i z \cdot d} \, dz}{\|f\|_2^2} \]

Therefore

\[ \left( \frac{1}{\|L\|_2^2} \int_{\mathbb{R}^n} |x|^2 |f(x)|^2 \, dx \right) \left( \frac{1}{\|L\|_2^2} \int_{\mathbb{R}^n} |y|^2 |L[f](y)|^2 \, dy \right) \]

\[ = \frac{b^2 (m+1)m}{4} \geq \frac{b^2 m^2}{4}. \]

Thus the unit energy complex paravector signal is indeed satisfied the uncertainty \textbf{Theorem 4.2}.

6. Conclusion

In this paper we developed the definition of Clifford LCT. The various properties of LCT such as partial derivative, Plancherel and Parseval theorems are established. Using the well-known Plancherel theorem, we established two uncertainty principles for hypercomplex signals in the LCT domains. The first principle (\textbf{Theorem 4.1}) states that the product of the variances of multivector- (complex paravector-) valued signals in the spatial and frequency domains has a lower bound. While the second principle (\textbf{Theorem 4.2}) states the directional case. It is shown that only a \(m\)-D Gaussian signal minimizes the uncertainties. With the help of these principles, they are useful in the time–frequency analysis, developing sampling theory in the \(m\)-D case, filter design, signal synthesis and optics.

We also note that the present theory can be generalized to complex Clifford- (\(\mathbb{C}_{0,m}\)) valued signals without difficulties. Further investigations on this topic are now under investigation and will be reported in a forthcoming paper.

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