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# Nonlinear difference equations for the generalized little $q$-Laguerre polynomials

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**ABSTRACT**

In this paper, a study is made on polynomials orthogonal with respect to the generalized little $q$-Laguerre weight, defined by $w(x) = x^{\alpha}(q^2x^2;q^2)_{\infty}$, $0 < x < 1$. Here $0 < q < 1$, $\alpha > 0$ and $(a;q)_{\infty} := \prod_{k=0}^{\infty}(1-a\,q^k)$. This weight is supported on the exponential lattice $\{1, q, q^2, \ldots, q^k, \ldots\}$. Let the subleading coefficient of $x^{n-1}$ of the monic polynomials be $\delta_n$. From the $q$-analogue of the ladder operators, the associated supplementary conditions and a 'sum rule', we deduce a system of difference equations satisfied by $\delta_n$. This system is used to obtain the first few terms in a formal asymptotic expansion of $\delta_n$. We express the recurrence coefficients in terms of this subleading coefficient and show that the first few terms in the formal expansions in powers of $q^n$ agree with the first few terms for the corresponding expansions of the recurrence coefficients in the classical case. Moreover, we find certain non-linear difference equations for the recurrence coefficients of the monic polynomials, auxiliary functions in the ladder operators and for $\delta_n$. We also observe the phenomenon of singularity confinement, related to that observed in the $q$-discrete Painlevé equations. Furthermore, we give a generalization of the weight function, characterized by $w(x/q)/w(x) = Ax^2 + Bx + C$ for $A \neq 0$, $C \neq 1$ on the exponential lattice. In this situation we find another system of difference equations satisfied by $\delta_n$ and study its behaviour for large $n$. The paper ends with the discussion on a deformation of the generalized little $q$-Laguerre weight.

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## 1. Introduction

Orthogonal polynomials appear naturally in various problems in classical mathematical physics, see for example [20,27]. They have many useful properties. They also provide a natural way to expand solutions to important ordinary and partial differential equations similar to the expansion of periodic functions in a linear combination of trigonometric functions in the Fourier series.

For example, the Hermite polynomials show up in the quantum mechanical treatment of Harmonic oscillators, and the Laguerre polynomials appear in the propagation of electromagnetic waves (see Section 4.25, [27]).

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From the definition of orthogonality, orthogonal polynomials can be generated by the Gram-Schmidt process [13,32]. As a consequence, one finds a sequence of polynomials that are pairwise orthogonal with respect to a measure, either discrete or continuous. In the Askey scheme [26] one finds a list of polynomials orthogonal with respect to either discrete or continuous measures, where the recurrence coefficients in the three-term recurrence relation (defined below in (1.4)) are obtained explicitly. There are numerous generalizations of the classical weights such that the corresponding recurrence coefficients are related to the solutions of either discrete or differential Painlevé equations (see, for instance, [14,28,33]). Associated with the theory of orthogonal polynomials is the classical moment problem [1]: for a given infinite set of moments to find a unique measure that characterizes them. Berg, Chen and Ismail [5] gave a new characterization: the classical moment problem is indeterminate if and only if the smallest eigenvalue of the Hankel (or moment) matrix of order $n$ is bounded away from 0, as $n$ tends to infinity.

In this paper we study monic polynomials orthogonal with respect to a class of discrete measures that appeared in [6,15]. The orthogonality condition reads

$$\int_a^b P_n(x)P_m(x)w(x)d_qx = \frac{\delta_{n,m}}{\gamma_n^2}, \tag{1.1}$$

and the $q$-Jackson integral [16] reads

$$\int_a^b f(x)d_qx = b(1-q)\sum_{n=0}^{\infty} q^n f(bq^n) - a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n). \tag{1.2}$$

Here $w(x)$ is the weight function supported on the exponential lattice $\{aq^k, bq^k | k = 0, 1, 2, \ldots\}$ and $\delta_{n,m}$ is the Kronecker delta.

Monic orthogonal polynomials

$$P_n(x) = x^n + \delta_n x^{n-1} + \cdots + P_n(0), \tag{1.3}$$

satisfy the three term recurrence relation [32]

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + a_n^2P_{n-1}(x), \quad n \geq 0. \tag{1.4}$$

Here $P_0(x) = 1$, $P_{-1}(x) = 0$ and $a_0 = 0$. From (1.3) we also have $\delta_0 = 0$, and $\delta_1 = -b_0$. The coefficients $a_n$ and $b_n$ in (1.4) are called the recurrence coefficients. We shall also call $\delta_n$ the subleading coefficient.

The orthonormal polynomials

$$\int_a^b p_m(x)p_n(x)w(x)d_qx = \delta_{m,n} \tag{1.5}$$

with $p_n(x) = \gamma_n P_n(x)$ satisfy a slightly different recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x). \tag{1.6}$$

We remark that from the three term recurrence relations (1.4) it follows immediately that

$$b_n = \delta_n - \delta_{n+1}. \tag{1.7}$$

We also have

$$a_n = \frac{\gamma_{n-1}}{\gamma_n}.$$

Recently, there have been numerous studies to derive nonlinear difference equations for recurrence coefficients and auxiliary quantities that appear in the definition of ladder operators for various weights (see, e.g. [3,8–11,22] and the references therein). The subleading coefficient is particularly useful in deriving Painlevé equations (e.g. [12]) for the recurrence coefficients. Moreover, nonlinear difference equations derived in this paper give examples of discrete integrable equations. There are many papers on integrability of difference equations of second and higher order, methods to detect integrability in equations and various applications (see, for instance, [2,4,18,19,23–25,29–31]).

Our main interest is in the determination of the recurrence coefficients $a_n$ and $b_n$ and of the subleading coefficient $\delta_n$ that appear in the monomial expansion of monic orthogonal polynomials for the generalized little $q$-Laguerre weight. We will first express the recurrence coefficients in terms of a number of auxiliary quantities. From the compatibility conditions and the sum rule we deduce a number of non-linear difference equations (Theorem 4.8, Theorem 5.5, Theorem 6.3), and ultimately a system of difference equations satisfied by $\delta_n$ (see Proposition 4.7 and Proposition 5.4). One of the equations in the systems is one of the discrete Painlevé equations and, therefore, it is natural to study the singularity confinement of the system. The main result of this paper are the second and third order nonlinear difference equations (of total degree 4) for the subleading coefficient $\delta_n$ (Theorem 4.8, Theorem 5.5, Theorem 6.3). The third order difference equations for $\delta_n$ are linear in $\delta_{n+2}$ and they can be used to calculate the subleading coefficient recursively (see Section 4.2). Since one of the recurrence coefficients, $b_n$, can be expressed in terms of $\delta_n$ as (1.7), we can calculate recursively this coefficient as well. The second order equations are quadratic in $\delta_{n+1}$. Note that Theorem 5.5 and Theorem 6.3 are for generalized weights, and, therefore, the coefficients in nonlinear difference equations are longer than in Theorem 4.8. We present them in Appendix 1. Theorem 6.3 is given for a very large class of weights with the assumption that Equation (6.1) holds, therefore it is of theoretical importance. Theorem 6.3 includes the previous theorems as particular cases. We present all theorems separately because of the previous studies [6,15]. As an application, from nonlinear difference equations we obtain a few terms of the formal asymptotic expansion of the subleading coefficient $\delta_n$ and of the recurrence coefficients (Section 4.2). We observe some similarity (in the first few terms) of such expansions for generalized weights with the expansions in the classical case. This deserves further detailed study.

The structure of this paper is as follows. In Section 2 we recall some facts from $q$-calculus. In Section 3 we review the definition and basic properties of ladder operators for monic $q$-orthogonal polynomials and for $q$-orthonormal polynomials on the exponential lattice and give the supplementary conditions that will further be used in the derivation of the relevant nonlinear difference equations. We illustrate the theory of ladder operators for

classical little $q$-Laguerre polynomials. In Section 4 we study in detail the generalized little $q$-Laguerre polynomials. In particular, we derive a system of nonlinear difference equations for $\delta_n$ and its behaviour for large $n$ (the first few terms in the formal asymptotic expansion). We also observe the phenomenon of the singularity confinement for a certain system of difference equations. In Section 5 we discuss orthogonal polynomials with respect to the family of positive weights defined by $w(x/q)/w(x) = Ax^2 + Bx + C$ with $A \neq 0,\ C \neq 1$ and $w(0) = w(1/q) = 0$. This family includes the generalized little $q$-Laguerre weight on the exponential lattice from the previous section. We also derive a system of difference equations for the subleading coefficient $\delta_n$ and discuss the asymptotic behaviour. Finally, in Section 6 we work on the deformation of a class of functions associated with the generalized little $q$-Laguerre weight, and obtain a system of difference equations for $\delta_n$.

## 2. $q$-Calculus

In this section we recall the main formulas from the $q$-calculus (for further details see [16,26]).

Let $D_q$ be a $q$-analogue of the difference operator,

$$(D_q f)(x) = \begin{cases} \frac{f(x)-f(qx)}{x(1-q)} & \text{if } x \neq 0, \\ f'(0) & \text{if } x = 0. \end{cases}$$

The $q$-product rule, which is frequently used, is as follows:

$$D_q(f(x)g(x)) = (D_q f(x))g(x) + f(xq)D_q g(x). \tag{2.1}$$

For the $q$-integral (1.2) we have the following analogue of integration by parts:

$$\int_a^b f(x)D_q g(x)d_q x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)D_q f(x)d_q x.$$

We also have

$$q\int_a^b f(qx)d_q x = \int_a^b f(x)d_q x + (1-q)(af(a) - bf(b)). \tag{2.2}$$

## 3. The ladder operators

The paper [7] presents ladder operators and differential equations for orthogonal polynomials with a continuous weight, later on the paper [22] shows that the $q$-orthogonal polynomials also satisfy the lowering ladder operator relation, which can be used to deduce useful compatibility conditions, and ultimately compute the recurrence coefficients.

Let $u$ be defined by

$$u(x) := -\frac{D_{q^{-1}} w(x)}{w(x)}. \tag{3.1}$$

Throughout this paper we assume that the weight is defined on the $q$-exponential lattice $\{aq^n,\ bq^n | n = 0, 1, \ldots\}$ and it vanishes at $a/q$ and $b/q$, i.e. $w(a/q) = w(b/q) = 0$. For the

monic $q$-orthogonal polynomials $P_n(x)$ we have the following relation:

$$D_q P_n(x) = a_n^2 A_n(x) P_{n-1}(x) - B_n(x) P_n(x) \tag{3.2}$$

with

$$A_n(x) = \gamma_n^2 \int_a^b \frac{u(qx) - u(y)}{qx - y} P_n(y) P_n(y/q) w(y) d_q y, \tag{3.3}$$

$$B_n(x) = \gamma_{n-1}^2 \int_a^b \frac{u(qx) - u(y)}{qx - y} P_n(y) P_{n-1}(y/q) w(y) d_q y. \tag{3.4}$$

We call (3.2) the lowering equation. Due to the assumption on the weight, there are no boundary terms in the formulas above. It was shown in [10] that the function $A_n(x)$ and $B_n(x)$ satisfy the supplementary conditions

$$B_{n+1}(x) + B_n(x) = (x - b_n) A_n(x) + x(q-1) \sum_{j=0}^{n} A_j(x) - u(qx), \tag{$qS_1$}$$

$$a_{n+1}^2 A_{n+1}(x) - a_n^2 A_{n-1}(x) = 1 + (x - b_n) B_{n+1}(x) - (qx - b_n) B_n(x). \tag{$qS_2$}$$

From $(qS_1)$ and $(qS_2)$, there is another identity, 'the sum rule', which is often helpful:

$$a_n^2 A_n(x) A_{n-1}(x) = B_n(x)^2 + u(qx) B_n(x) + (1 + (1-q) x B_n(x)) \sum_{j=0}^{n-1} A_j(x), \tag{$qS_2'$}$$

see [11] for the proof.

By eliminating $P_{n-1}(x)$ between (3.2) and (1.4) we get the following raising equation:

$$D_q P_{n-1}(x) = ((x - b_{n-1}) A_{n-1} - B_{n-1}) P_{n-1}(x) - A_{n-1} P_n(x). \tag{3.5}$$

The lowering and raising equations allow us to obtain a second order difference equation for $P_n(x)$. Indeed, let us set

$$L_{1,n} = B_n + D_q, \tag{3.6}$$

$$L_{2,n} = (x - b_{n-1}) A_{n-1} - B_{n-1} - D_q. \tag{3.7}$$

Then

$$L_{1,n} P_n = a_n^2 A_n P_{n-1}, \tag{3.8}$$

$$L_{2,n} P_{n-1} = A_{n-1} P_n. \tag{3.9}$$

Now by combining the two equations we get the second-order $q$-difference equation

$$L_{2,n} \left( \frac{1}{A_n} L_{1,n} P_n \right) = a_n^2 A_{n-1} P_n, \tag{3.10}$$

which is of the form

$$D_q^2 P_n(x) + R_n(x) D_q P_n(x) + S_n(x) P_n(x) = 0 \tag{3.11}$$

with

$$R_n(x) = B_n(qx) - \frac{D_q A_n(x)}{A_n(x)} + \frac{A_n(qx)}{A_n(x)}(B_{n-1}(x) - (x - b_{n-1})A_{n-1}(x)), \tag{3.12}$$

$$\begin{aligned} S_n(x) &= a_n^2 A_n(qx) A_{n-1}(x) + D_q B_n(x) - \frac{B_n(x)}{A_n(x)} D_q A_n(x) \\ &\quad + B_n(x) \frac{A_n(qx)}{A_n(x)}(B_{n-1}(x) - (x - b_{n-1})A_{n-1}(x)). \end{aligned} \tag{3.13}$$

For future reference, we list below analogous formulas in the orthonormal case. The $q$-orthonormal polynomials $p_n(x)$ satisfy the following relation:

$$D_q p_n(x) = \tilde{A}_n(x) p_{n-1}(x) - \tilde{B}_n(x) p_n(x)$$

with

$$\tilde{A}_n(x) = a_n \int_a^b \frac{u(qx) - u(y)}{qx - y} p_n(y) p_n(y/q) w(y) d_q y, \tag{3.14}$$

$$\tilde{B}_n(x) = a_n \int_a^b \frac{u(qx) - u(y)}{qx - y} p_n(y) p_{n-1}(y/q) w(y) d_q y. \tag{3.15}$$

The following compatibility conditions hold:

$$\begin{aligned} \tilde{B}_{n+1}(x) + \tilde{B}_n(x) &= (x - b_n)\frac{\tilde{A}_n(x)}{a_n} \\ &\quad + x(q-1)\sum_{j=0}^{n} \frac{\tilde{A}_j(x)}{a_j} - u(qx), \end{aligned} \tag{3.16}$$

$$a_{n+1}\tilde{A}_{n+1}(x) - a_n^2 \frac{\tilde{A}_{n-1}(x)}{a_{n-1}} = 1 + (x - b_n)\tilde{B}_{n+1}(x) - (qx - b_n)\tilde{B}_n(x). \tag{3.17}$$

It is not difficult to show that

$$\tilde{A}_n = a_n A_n, \quad \tilde{B}_n = B_n.$$

The following proposition holds true.

**Proposition 3.1:** *Let $\tilde{A}_n(x)$ and $\tilde{B}_n(x)$ be given by (3.14) and (3.15). Then*

$$\begin{aligned} \frac{a_n}{a_{n-1}} \tilde{A}_n(x) \tilde{A}_{n-1}(x) &= \tilde{B}_n(x)^2 + u(qx)\tilde{B}_n(x) \\ &\quad + (1 + (1-q)x\tilde{B}_n(x)) \sum_{j=0}^{n-1} \frac{\tilde{A}_j(x)}{a_j}. \end{aligned} \tag{3.18}$$

**Example 3.2:** The classical little $q$-Laguerre polynomials are orthogonal with respect to the weight

$$w(x) = x^{\alpha}(qx; q)_{\infty}, \quad 0 < x < 1, \quad \alpha > 0$$

on the $q$-lattice $\{q^k | k = 0, 1, \ldots\}$. Here $(x; q)_{\infty}$ is the $q$-Pochhammer symbol defined by $(x; q)_{\infty} := \prod_{k=0}^{\infty}(1 - xq^k)$. It can easily be computed that

$$u(x) = \frac{q^{1-\alpha}(1 - q^{\alpha} - x)}{(q-1)x}.$$

Substituting

$$A_n = \frac{R_n}{(1-q)x}, \quad B_n = \frac{r_n}{(1-q)x},$$

where

$$R_n = \gamma_n^2(q^{-\alpha} - 1)\int_0^1 P_n(y)P_n(y/q)\frac{w(y)}{y}d_q y,$$

$$r_n = \gamma_{n-1}^2(q^{-\alpha} - 1)\int_0^1 P_n(y)P_{n-1}(y/q)\frac{w(y)}{y}d_q y$$

into $(qS_1)$ and $(qS_2)$ and collecting the coefficients at $x$, we get a system of equations for the recurrence coefficients and auxiliary quantities $R_n$ and $r_n$. The system can easily be solved to give:

$$R_n = q^{-\alpha-n}, \quad r_n = q^n - 1,$$

$$b_n = q^n(1 + q^{\alpha} - q^{\alpha+n}(1+q)), \quad a_n^2 = q^{2n+\alpha-1}(1-q^n)(1-q^{\alpha+n}).$$

Hence, by using (1.7), we get

$$\delta_n = \frac{(q^n - 1)(q^{\alpha+n} - 1)}{q - 1}.$$

## 4. Generalized little $q$-Laguerre weight

In this section we consider the weight studied in [6] and deduce new difference equations involving the recurrence coefficients and $\delta_n$. Moreover, we shall obtain the first few terms in the asymptotic expansion of $\delta_n$ for large $n$.

The weight function that we shall consider in this section is given by

$$w(x) = x^{\alpha}(q^2x^2; q^2)_{\infty}, \quad \alpha > 0, \tag{4.1}$$

where $0 < q < 1$ and

$$(x; q)_{\infty} := \prod_{k=0}^{\infty}(1 - xq^k).$$

The weight is supported on the exponential lattice $\{q^k | k = 0, 1, 2, \ldots\}$ and it satisfies $w(0) = w(1/q) = 0$.

An easy computation shows that the function $u(x)$ used in the definition of ladder operators reads

$$u(x) = \frac{1}{1-q}\left(\frac{q-q^{1-\alpha}}{x} + q^{1-\alpha}x\right). \tag{4.2}$$

From [10] we have the following identities to be used later on:

$$\int_0^1 u(y)P_n(y)P_n(y/q)w(y)d_qy = 0, \tag{4.3}$$

$$\int_0^1 u(y)P_n(y)P_{n-1}(y/q)w(y)d_qy = \frac{q}{\gamma_{n-1}^2}\frac{1-q^n}{1-q}. \tag{4.4}$$

Moreover, by using orthogonality, we have

$$\int_0^1 P_n(y)P_n(y/q)w(y)d_qy = \frac{1}{q^n\gamma_n^2}, \tag{4.5}$$

$$\int_0^1 P_n(y)P_{n-1}(y/q)w(y)d_qy = 0. \tag{4.6}$$

The functions $A_n$ and $B_n$, defined by (3.3) and (3.4) in the lowering ladder operator relation, read [6]

$$A_n(x) = \frac{R_n}{(1-q)x} + \frac{q^{-n-\alpha+1}}{1-q}, \tag{4.7}$$

$$B_n(x) = \frac{r_n}{(1-q)x}, \tag{4.8}$$

where

$$R_n = \gamma_n^2(q^{-\alpha}-1)\int_0^1 P_n(y)P_n(y/q)\frac{w(y)}{y}d_qy, \tag{4.9}$$

$$r_n = \gamma_{n-1}^2(q^{-\alpha}-1)\int_0^1 P_n(y)P_{n-1}(y/q)\frac{w(y)}{y}d_qy. \tag{4.10}$$

**Lemma 4.1 [6]:** *The following relations hold true:*

$$r_{n+1} + r_n = -b_n R_n - (1 - q^{-\alpha}), \tag{4.11}$$

$$R_n - b_n q^{-n-\alpha+1} - (1-q)\sum_{j=0}^{n} R_j = 0, \tag{4.12}$$

$$a_n^2 = q^{n+\alpha-1}(r_n + 1 - q^n), \tag{4.13}$$

$$a_n^2 R_n R_{n-1} = r_n(r_n + 1 - q^{-\alpha}), \tag{4.14}$$

$$R_n^2 = -q^{-2n-\alpha}((r_n+1)(r_{n+1}+1) - q^{-\alpha}), \tag{4.15}$$

$$\begin{aligned} r_n &\left( qR_n + R_{n-1} - (1-q)\sum_{j=0}^{n-1} R_j \right) \\ &= (1-q^{-n})(q^{n+1}R_n + q^n R_{n-1}) + (1-q)\sum_{j=0}^{n-1} R_j, \end{aligned} \tag{4.16}$$

$$b_n(1+r_n)q^{-n-\alpha+1} = q^{n+1}R_n - R_{n-1}(r_n + 1 - q^n), \tag{4.17}$$

$$\begin{aligned} &q^{-n-\alpha+1}(r_n+1)(r_{n+1}+r_n+1-q^{-\alpha}) \\ &\quad = -q^{n+1}R_n^2 + R_n R_{n-1}(r_n + 1 - q^n). \end{aligned} \tag{4.18}$$

In [6, Theorem 1.3] it is shown that the recurrence coefficients of the orthogonal polynomials for the weight (4.1) on the exponential lattice $\{q^k | k = 0, 1, 2, \ldots\}$ are given for $n \geq 1$ by

$$a_n^2 = q^{n+\alpha/2-1}(x_n - q^{n+\alpha/2}),$$

and

$$\begin{aligned} b_n^2 x_n^2 q^{-2n-\alpha} = 1 - x_n x_{n+1} - q^{-2n}(x_n x_{n-1} - 1)(x_n q^{-\alpha/2} - q^n)^2 \\ - 2(x_n - q^{\alpha/2})(x_n - q^{-\alpha/2}), \end{aligned} \tag{4.19}$$

with $b_0^2 = 1 - q^{\alpha/2}x_1$, where $x_n$ satisfies the $q$-discrete Painlevé V equation

$$(x_n x_{n-1} - 1)(x_n x_{n+1} - 1) = \frac{q^{2n+\alpha}(x_n - q^{\alpha/2})^2(x_n - q^{-\alpha/2})^2}{(x_n - q^{n+\alpha/2})^2} \tag{4.20}$$

with initial values $x_0 = q^{\alpha/2}$ and $x_1 = q^{-\alpha/2}(1 - \mu_1^2/\mu_0^2)$, where $\mu_0$ and $\mu_1$ are the first two moments of the weight $w$. Here

$$r_n = q^{-\frac{\alpha}{2}} x_n - 1. \tag{4.21}$$

To get Equation (4.20), one needs to use Equations (4.13), (4.14) and (4.15) from Lemma 4.1. Using

$$R_nR_{n-1} = \frac{r_n(r_n+1-q^{-\alpha})}{q^{n+\alpha-1}(r_n+1-q^n)}, \tag{4.22}$$

$$R_n^2 = -q^{-2n-\alpha}((r_n+1)(r_{n+1}+1)-q^{-\alpha}), \tag{4.23}$$

Equation (4.20) is then equivalent to

$$f_n^2 - g_ng_{n-1} = 0, \tag{4.24}$$

where $f_n = R_nR_{n-1}$ and $g_n = R_n^2$. Noting (4.21), the following equations hold:

$$f_n = \frac{(x_n - q^{\alpha/2})(x_n - q^{-\alpha/2})}{q^{n+\frac{3\alpha}{2}-1}(x_n - q^{n+\frac{\alpha}{2}})},$$

$$g_n = q^{-2n-2\alpha}(1 - x_nx_{n+1})$$

and from (4.13) one can get (4.19).

### 4.1. Nonlinear difference equations

The main objective of this subsection is to derive nonlinear difference equations for $\delta_n$ and to express other quantities, like the recurrence coefficients, in terms of $\delta_n$. Moreover, we find new expressions of $b_n^2$ in terms of $x_n$, which leads to new higher order difference equations satisfied by $x_n$. We also show how to calculate $b_n$ instead of $b_n^2$ directly.

**Lemma 4.2:** *The auxiliary quantity $R_n$ defined by (4.9) can be expressed in terms of $\delta_n$ as follows:*

$$R_n = q^{1-n-\alpha}(\delta_n - \delta_{n+1}/q), \tag{4.25}$$

*Consequently,*

$$\sum_{j=0}^{n} R_j = -q^{-n-\alpha}\delta_{n+1}. \tag{4.26}$$

***Proof*:** Substituting $u(x)$ given by (4.2) into (4.3), we get

$$\int_0^1 P_n(y)P_n(y/q)\frac{w(y)}{y}d_qy = \frac{q^{1-\alpha}}{q(q^{-\alpha}-1)}\int_0^1 yP_n(y)P_n(y/q)w(y)d_qy. \tag{4.27}$$

To evaluate the integral above, we use the identity

$$y^{n+1} = P_{n+1}(y) - \delta_{n+1}y^n + \dots. \tag{4.28}$$

Thus,

$$\begin{aligned} yP_n(y/q) &= \frac{y^{n+1}}{q^n} + \delta_n \frac{y^n}{q^{n-1}} + \dots \\ &= (P_{n+1}(y) - \delta_{n+1}y^n + \dots)/q^n + \delta_n y^n/q^{n-1} + \dots \\ &= P_{n+1}(y)/q^n + (\delta_n/q^{n-1} - \delta_{n+1}/q^n)y^n + \dots \\ &= P_{n+1}(y)/q^n + (\delta_n/q^{n-1} - \delta_{n+1}/q^n)P_n(y) + \dots . \end{aligned} \tag{4.29}$$

Finally,

$$\begin{aligned} &\int_0^1 yP_n(y)P_n(y/q)w(y)d_qy \\ &= \int_0^1 P_n(y)(P_{n+1}(y)/q^n + (\delta_n/q^{n-1} - \delta_{n+1}/q^n)P_n(y) + \dots)w(y)d_qy, \\ &= \frac{1}{\gamma_n^2}(\delta_n/q^{n-1} - \delta_{n+1}/q^n). \end{aligned} \tag{4.30}$$

From (4.9), (4.27) and (4.30) the first relation (4.25) in the lemma is proved.

To evaluate the sum of $R_n$, we substitute (4.25) and $b_n = \delta_n - \delta_{n+1}$ into (4.12) and obtain (4.26). Clearly, formula (4.26) is immediate as a telescopic sum of (4.25). □

**Remark 1:** Substituting $u(x)$ into (4.4), we find

$$\begin{aligned} &\int_0^1 P_n(y)P_{n-1}(y/q)\frac{w(y)}{y}d_qy \\ &= \frac{q^n - 1}{\gamma_{n-1}^2(q^{-\alpha} - 1)} + \frac{q^{-\alpha}}{(q^{-\alpha} - 1)}\int_0^1 yP_n(y)P_{n-1}(y/q)w(y)d_qy \\ &= \frac{q^n - 1}{\gamma_{n-1}^2(q^{-\alpha} - 1)} + \frac{q^{-\alpha}}{(q^{-\alpha} - 1)}\int_0^1 (P_{n+1}(y) + b_nP_n(y) \\ &\quad + a_n^2P_{n-1}(y))P_{n-1}(y/q)w(y)d_qy \\ &= \frac{q^n - 1}{\gamma_{n-1}^2(q^{-\alpha} - 1)} + \frac{q^{-\alpha}}{(q^{-\alpha} - 1)}a_n^2\gamma_{n-1}^{-2}q^{-n+1}. \end{aligned} \tag{4.31}$$

The second equality follows from the recurrence relation of $P_n(x)$, then we use orthogonality to obtain the third equality. Finally, from Equations (4.10) and (4.31) we find (4.13).

Next we find formulas in addition to those given in Lemma 4.1.

**Lemma 4.3:** *The auxiliary quantities $R_n$ and $r_n$, defined by (4.9) and (4.10) respectively, and the recurrence coefficient $b_n$ in (1.4) satisfy the following relations:*

$$b_{n-1}(1 + r_n)q^{-n-\alpha+1} = q^{n-1}R_{n-1} - R_n(r_n + 1 - q^n), \tag{4.32}$$

$$q^{-n-\alpha+1}(r_n + 1)(r_n + r_{n-1} + 1 - q^{-\alpha}) = -q^{n-1}R_{n-1}^2 + R_nR_{n-1}(r_n + 1 - q^n). \tag{4.33}$$

***Proof*:** We replace $n$ by $n-1$ in Equation (4.12), that is,

$$(1-q)\sum_{j=0}^{n-1} R_j = R_{n-1} - b_{n-1}q^{-n-\alpha+2}, \tag{4.34}$$

and substitute the sum $\sum_{j=0}^{n-1} R_j$ given by (4.34) into (4.16) to find

$$r_n(qR_n + b_{n-1}q^{-n-\alpha+2}) = R_{n-1} - b_{n-1}q^{-n-\alpha+2} + (1-q^{-n})(q^{n+1}R_n + q^nR_{n-1}). \tag{4.35}$$

Dividing by $q$ on both sides of the above equation, we find

$$r_n(R_n + b_{n-1}q^{-n-\alpha+1}) = -b_{n-1}q^{-n-\alpha+1} + q^nR_n - R_n + q^{n-1}R_{n-1}, \tag{4.36}$$

which gives (4.32) after the re-arrangement of terms. Next, we multiply both sides of this equation by $-R_{n-1}$ and obtain

$$-b_{n-1}R_{n-1}(1+r_n)q^{-n-\alpha+1} = -q^{n-1}R_{n-1}^2 + R_nR_{n-1}(r_n + 1 - q^n). \tag{4.37}$$

Replacing $n$ by $n-1$ in (4.11) gives

$$-b_{n-1}R_{n-1} = r_n + r_{n-1} + (1-q^{-\alpha}), \tag{4.38}$$

and substituting $-b_{n-1}R_{n-1}$ from (4.38) into (4.37) gives

$$q^{-n-\alpha+1}(r_n+1)(r_n + r_{n-1} + 1 - q^{-\alpha}) = -q^{n-1}R_{n-1}^2 + R_nR_{n-1}(r_n + 1 - q^n).$$

Note that Equation (4.32) is different from (4.17) and Equation (4.33) is different from (4.18). □

**Remark 2:** Substituting $A_n(x)$ given by (4.7), $B_n(x)$ given by (4.8) and $u(x)$ given by (4.2) into Equation $(qS_2')$, we find, after collecting the coefficients in $x$, the following equations: Equations (4.13), (4.14) and

$$a_n^2 = \frac{q^{n+\alpha-1}(r_n+1)(1-q)S_{n-1}}{qR_n + R_{n-1}}, \tag{4.39}$$

where $S_{n-1} = \sum_{j=0}^{n-1} R_j$. Substituting $(1-q)\sum_{j=0}^{n-1} R_j$ given by (4.34) into the equation above, we get

$$a_n^2 = \frac{q^{n+\alpha-1}(r_n+1)(R_{n-1} - b_{n-1}q^{-n-\alpha+2})}{qR_n + R_{n-1}}. \tag{4.40}$$

Since $a_n^2$ is also given by (4.13), equating these two equations, we get the following equation for $b_{n-1},\ R_n,\ r_n$:

$$b_{n-1}(1+r_n)q^{-n-\alpha+1} = q^{n-1}R_{n-1} - R_n(r_n + 1 - q^n), \tag{4.41}$$

which is the same as (4.32).

We have the following alternative to (4.19) expressions of $b_n^2$ in terms of $x_n$ (or $r_n$ via (4.21)).

**Proposition 4.4:** *The recurrence coefficient $b_n$ of polynomials orthogonal with respect to the discrete weight (4.1) supported on the exponential lattice $\{1,\ q,\ q^2,\ \ldots\}$ can be expressed in terms of $x_n$ satisfying (4.20) as follows:*

(i)

$$b_{n-1}^2 = \frac{\left(x_n^2 x_{n+1} - q^{n+\frac{\alpha}{2}}\left(q^n - 2 + q^{n+\alpha} - q^{n+\frac{\alpha}{2}} x_{n+1}\right) - x_n\left(1 - q^{2n+\alpha} + 2q^{n+\frac{\alpha}{2}} x_{n+1}\right)\right)^2}{q^2\left(q^{n+\frac{\alpha}{2}} - x_n\right)^2 (1 - x_n x_{n+1})}, \tag{4.42}$$

(ii)

$$b_n^2 = \frac{q^n\left(q^{n+\frac{\alpha}{2}} - x_n\right)\left(q^n + q^{n+\alpha} - 1 + x_{n-1}x_n - q^{n+\frac{\alpha}{2}}(x_{n-1} + x_n)\right)\left(q^\alpha + 1 - q^{\frac{\alpha}{2}}(x_n + x_{n+1})\right)^2}{\left(q^{\frac{\alpha}{2}} - x_n\right)\left(q^{\frac{\alpha}{2}} x_n - 1\right)\left(q^n + q^{n+\alpha} - 1 + x_n x_{n+1} - q^{n+\frac{\alpha}{2}}(x_n + x_{n+1})\right)}, \tag{4.43}$$

(iii)

$$\begin{aligned} b_n^2 x_{n+1}^2 q^{-2n-\alpha} &= 1 - x_n x_{n+1} - q^{-2n-2}(x_{n+1}x_{n+2} - 1)(x_{n+1}q^{-\frac{\alpha}{2}} - q^{n+1})^2 \\ &\quad - 2(x_{n+1} - q^{\frac{\alpha}{2}})(x_{n+1} - q^{\frac{-\alpha}{2}}), \end{aligned} \tag{4.44}$$

(iv)

$$\begin{aligned} b_n^2 x_n^2 q^{-2n-\alpha} &= 1 - x_n x_{n+1} - q^{-2n}(x_n x_{n-1} - 1)(x_n q^{-\alpha/2} - q^n)^2 \\ &\quad - 2(x_n - q^{\alpha/2})(x_n - q^{-\alpha/2}), \end{aligned} \tag{4.45}$$

(v)

$$b_n^2 = \frac{(q^{n+\frac{\alpha}{2}}(2 - q^n - q^{n+\alpha}) + x_{n-1}(x_n - q^{n+\frac{\alpha}{2}})^2 + x_n(q^{2n+\alpha} - 1))^2}{(x_n - q^{n+\frac{\alpha}{2}})^2(1 - x_n x_{n-1})}, \tag{4.46}$$

(vi)

$$b_n^2 = \frac{q^{2n}(1 + q^\alpha - q^{\frac{\alpha}{2}} x_n - q^{\frac{\alpha}{2}} x_{n+1})^2}{1 - x_n x_{n+1}}. \tag{4.47}$$

***Proof:*** (i) First we multiply Equation (4.32) by $R_n$ and replace $R_n R_{n-1}$ from (4.22). This gives a quadratic equation in $R_n$:

$$(r_n + 1 - q^n)R_n^2 + b_{n-1}(1 + r_n)q^{1-n-\alpha}R_n - \frac{r_n(r_n + 1 - q^{-\alpha})}{q^\alpha(r_n + 1 - q^n)} = 0. \tag{4.48}$$

Next we replace $R_n^2$ by (4.23) and obtain a linear equation in $R_n$, which can be solved as

$$R_n = \frac{r_n(r_n+1-q^{-\alpha}) + ((r_n+1)(r_{n+1}+1) - q^{-\alpha})(r_n+1-q^n)^2 q^{-2n}}{(r_n+1-q^n)(r_n+1)q^{1-n}b_{n-1}}. \quad (4.49)$$

Inserting $R_n$ given above into Equation (4.48), and using (4.21), we get an expression for $b_{n-1}^2$ in terms of $x_n$ and $x_{n+1}$ (4.42).

(ii) First we use Equation (4.12) and subtract from it the same equation with $n$ replaced by $n-1$. This leads to

$$b_n - b_{n-1}q - q^{n+\alpha-1}(qR_n - R_{n-1}) = 0. \quad (4.50)$$

Next we rewrite Equation (4.11) as

$$b_n = -\frac{(r_{n+1}+r_n+1-q^{-\alpha})}{R_n}, \quad (4.51)$$

substitute the expression of $b_n$ and $b_{n-1}$ obtained from (4.51) into (4.50) and replace $R_{n-1}$ from Equation (4.22). Finally, we get

$$R_n^2 = \frac{q^{-n-\alpha}r_n(r_n+1-q^{-\alpha})(r_{n+1}(r_n+1-q^n) + (r_n+1-q^{-\alpha})(1-q^n))}{(r_n+1-q^n)(r_{n-1}(r_n+1-q^n) + (r_n+1-q^{-\alpha})(1-q^n))}, \quad (4.52)$$

which expresses $R_n^2$ in terms of $r_n$, $r_{n+1}$ and $r_{n-1}$. To get (4.43), we substitute (4.52) into

$$b_n^2 = \frac{(r_{n+1}+r_n+1-q^{-\alpha})^2}{R_n^2}, \quad (4.53)$$

then using $r_n = q^{-\frac{\alpha}{2}}x_n - 1$, we get an expression for $b_n^2$ in terms of $x_{n-1}, x_n$, and $x_{n+1}$ (4.43).

(iii) Using (4.21), Equations (4.22) and (4.23) become

$$R_nR_{n-1} = \frac{(q^{-\frac{\alpha}{2}}x_n - 1)(q^{-\frac{\alpha}{2}}x_n - q^{-\alpha})}{q^{n+\alpha-1}(q^{-\frac{\alpha}{2}}x_n - q^n)}, \quad (4.54)$$

$$R_n^2 = -q^{-2n-2\alpha}(x_nx_{n+1} - 1). \quad (4.55)$$

Squaring both sides of Equation (4.32) and substituting (4.54) and (4.55) into the 'squared' equation, we find (4.44) after re-arranging and replacing $n-1$ by $n$.

(iv) This equation is Equation (4.19) found in [6].

(v), (vi) Equations (4.42)–(4.45) can be transformed into (4.46) and (4.47) by substituting $x_{n-1}$ or $x_{n+1}$ from qPV (4.20) one or several times with $n$ replace by $n-1$ or $n+1$. □

Note that Proposition 4.4 gives many difference equations for $x_n$. For instance, we can combine (4.52) with (4.23) and use (4.21), or equate different expressions for $b_n^2$ (also using (4.19)) in various combinations.

**Theorem 4.5:**

(i) *The following third order equation for $x_n$ holds:*

$$\begin{aligned}&(q^{n+\frac{\alpha}{2}+1}-x_{n+1})^2x_nx_{n+2}\\&\quad=q^{2n+\alpha+2}x_{n+1}^2-(2q^{n+\frac{\alpha}{2}+2}(q^{n+\alpha}+q^n-1)\\&\qquad-q^2x_{n-1}(q^{\frac{\alpha}{2}+n}-x_n)^2+(q^2-1)x_n)x_{n+1}\\&\qquad-q^{2n+\alpha+2}x_n^2+2q^{n+\frac{\alpha}{2}+1}(q^{n+\alpha+1}+q^{n+1}-1)x_n.\end{aligned}\tag{4.56}$$

(ii) *The following second order equation for $x_n$ holds:*

$$\begin{aligned}&(q^{n+\frac{\alpha}{2}}-x_n)^2(x_{n-1}x_n-1)x_{n+1}\\&\quad=q^{2n+\alpha}x_n^3-2q^{2n+\frac{\alpha}{2}}(1+q^\alpha)x_n^2+(4q^{2n+\alpha}+q^{2n+2\alpha}+q^{2n}-1)x_n\\&\qquad+x_{n-1}(q^{n+\frac{\alpha}{2}}-x_n)^2-2q^{n+\frac{\alpha}{2}}(q^{n+\alpha}+q^n-1).\end{aligned}\tag{4.57}$$

***Proof*:**

(i) Combining (4.44) with (4.45), we obtain Equation (4.56).

(ii) Combining (4.42) with (4.44), (4.44) with (4.47), (4.45) with (4.46), (4.45) with (4.47), we obtain Equation (4.57).

Similarly, we can study other combinations.

There are also two equations for $R_n^2$, namely,

$$R_n^2=\frac{q^{-n-\alpha}r_n(r_n+1-q^{-\alpha})(r_{n+1}(r_n+1-q^n)+(r_n+1-q^{-\alpha})(1-q^n))}{(r_n+1-q^n)(r_{n-1}(r_n+1-q^n)+(r_n+1-q^{-\alpha})(1-q^n))}.\tag{4.58}$$

$$R_n^2=-q^{-2n-\alpha}((r_n+1)(r_{n+1}-1)-q^{-\alpha}).\tag{4.59}$$

Equating the right-hand sides of these equations, and using $r_n=q^{-\frac{\alpha}{2}}x_n-1$, we get Equation (4.57).

□

Finally, we obtain nonlinear difference equations for $\delta_n$ and express the recurrence coefficients in terms if it.

**Proposition 4.6:** *The auxiliary quantity $r_n$ defined by (4.10) can be expressed in terms of $\delta_n$ as follows:*

$$r_n=\frac{q(q^n-1)\delta_{n-1}+(q-1)q^n\delta_n-(q^n-1)\delta_{n+1}}{q\delta_{n-1}-\delta_{n+1}}.\tag{4.60}$$

*The recurrence coefficients $a_n^2$ and $b_n$ are given in terms of $\delta_n$ by*

$$a_n^2=\frac{(q-1)q^{2n+\alpha-1}\delta_n}{q\delta_{n-1}-\delta_{n+1}},\quad b_n=\delta_n-\delta_{n+1}.\tag{4.61}$$

***Proof*:** The first equation is obtained by substituting $R_n$ from (4.25) and $b_n=\delta_n-\delta_{n+1}$ into (4.32). Substituting $r_n$ from (4.60) into (4.14) gives (4.61). □

**Proposition 4.7:** *The subleading coefficient $\delta_n$ of monic orthogonal polynomials for the weight (4.1) and the solutions $x_n$ of the q-discrete Painlevé V Equation (4.20) satisfy a system of second order difference equations given by*

$$\delta_{n+1} = \frac{q(q^{n+\frac{\alpha}{2}} - x_n)\delta_{n-1} + (q-1)q^{n+\frac{\alpha}{2}}\delta_n}{q^{n+\frac{\alpha}{2}} - x_n}, \tag{4.62}$$

$$(x_n x_{n-1} - 1)(x_n x_{n+1} - 1) = \frac{q^{2n+\alpha}(x_n - q^{\alpha/2})^2(x_n - q^{-\alpha/2})^2}{(x_n - q^{n+\alpha/2})^2}. \tag{4.63}$$

*Moreover,*

$$x_{n+1} = -x_n - q^{-n-\frac{\alpha}{2}}(\delta_n - \delta_{n+1})(q\delta_n - \delta_{n+1}) + q^{\frac{\alpha}{2}} + q^{-\frac{\alpha}{2}}, \tag{4.64}$$

*with initial values $x_0 = q^{\alpha/2}$ and $x_1 = q^{-\alpha/2}(1 - \mu_1^2/\mu_0^2)$, where $\mu_0$ and $\mu_1$ are the first two moments of the weight $w$.*

***Proof:*** The first Equation (4.62) follows from (4.60), i.e.

$$\delta_{n+1} = \frac{q(q^n - 1 - r_n)\delta_{n-1} + q^n(q-1)\delta_n}{q^n - 1 - r_n} \tag{4.65}$$

and from (4.21). Equation (4.63), as mentioned earlier, was derived in [6].

Equation (4.64) follows from (4.11) by using (4.25) and $b_n = \delta_n - \delta_{n+1}$. □

In the following theorem we shall present two difference equations which are satisfied by the subleading coefficient $\delta_n$.

**Theorem 4.8:** *The following equations hold for the subleading coefficient $\delta_n$ of monic orthogonal polynomials with respect to the weight (4.1):*

(i) *the third order difference equation, which turns out to be an algebraic equation of total degree 4 in $\delta_{n-1}$, $\delta_n$, $\delta_{n+1}$ and $\delta_{n+2}$, given by*

$$\sum_{p=0}^{1}\sum_{q=0}^{3}\sum_{r=0}^{3}\sum_{s=0}^{1} c_{p,q,r,s}\delta_{n-1}^p\delta_n^q\delta_{n+1}^r\delta_{n+2}^s = 0, \quad p+q+r+s \le 4, \tag{4.66}$$

*with 20 non-zero coefficients $c_{p,q,r,s}$*

$$\begin{aligned}
&c_{1,1,0,0} = q^{2+n} + q^{2+n+\alpha} - q^{2+2n+\alpha} - q^{3+2n+\alpha},\\
&c_{0,2,0,0} = q^{1+2n+\alpha} - q^{2+2n+\alpha}, \quad c_{1,3,0,0} = -q^3, \quad c_{1,0,1,0} = q^{2+2n+\alpha} - q^{3+2n+\alpha},\\
&c_{0,1,1,0} = -q^{1+n} - q^{1+n+\alpha} + q^{1+2n+\alpha} + q^{2+2n+\alpha},\\
&c_{1,2,1,0} = q^2 + q^3, \quad c_{0,3,1,0} = q^2, \quad c_{0,0,2,0} = -q^{1+2n+\alpha} + q^{2+2n+\alpha},\\
&c_{1,1,2,0} = -q^2, \quad c_{0,2,2,0} = -q - q^2, \quad c_{0,1,3,0} = q,\\
&c_{1,0,0,1} = -q^{1+n} - q^{1+n+\alpha} + q^{1+2n+\alpha} + q^{2+2n+\alpha},\\
&c_{0,1,0,1} = -q^{2n+\alpha} + q^{1+2n+\alpha}, \quad c_{1,2,0,1} = q^2,\\
&c_{0,0,1,1} = q^n + q^{n+\alpha} - q^{2n+\alpha} - q^{1+2n+\alpha}, \quad c_{1,1,1,1} = -q - q^2,\\
&c_{0,2,1,1} = -q, \quad c_{1,0,2,1} = q, \quad c_{0,1,2,1} = 1 + q, \quad c_{0,0,3,1} = -1;
\end{aligned}$$

(ii) *the second order difference equation, which turns out to be an algebraic equation of total degree 4 in* $\delta_{n-1}$, $\delta_n$ *and* $\delta_{n+1}$, *given by*

$$\sum_{p=0}^{2}\sum_{q=0}^{3}\sum_{r=0}^{2} c_{p,q,r}\delta_{n-1}^{p}\delta_n^{q}\delta_{n+1}^{r} = 0, \quad p+q+r \le 4, \tag{4.67}$$

*with 13 non-zero coefficients* $c_{p,q,r}$

$$\begin{aligned}
&c_{2,0,0} = -q^2 + q^{2+n} + q^{2+n+\alpha} - q^{2+2n+\alpha},\\
&c_{1,1,0} = -q^{1+n} + q^{2+n} - q^{1+n+\alpha} + q^{2+n+\alpha} + 2q^{1+2n+\alpha} - 2q^{2+2n+\alpha},\\
&c_{0,2,0} = -q^{2n+\alpha} + 2q^{1+2n+\alpha} - q^{2+2n+\alpha}, \quad c_{2,2,0} = -q^3 + q^4, \quad c_{1,3,0} = q^2 - q^3,\\
&c_{1,0,1} = 2q - 2q^{1+n} - 2q^{1+n+\alpha} + 2q^{1+2n+\alpha},\\
&c_{0,1,1} = q^n - q^{1+n} + q^{n+\alpha} - q^{1+n+\alpha} - 2q^{2n+\alpha} + 2q^{1+2n+\alpha},\\
&c_{2,1,1} = q^2 - q^3, \quad c_{1,2,1} = -q + 2q^2 - q^3, \quad c_{0,3,1} = -q + q^2,\\
&c_{0,0,2} = -1 + q^n + q^{n+\alpha} - q^{2n+\alpha}, \quad c_{1,1,2} = -q + q^2, \quad c_{0,2,2} = 1 - q.
\end{aligned}$$

***Proof*:** To derive Equation (4.66), we start from equations that can be found in [6], namely Equations (5.3a), (5.4a) and (5.4b) in [6]. They are Equations (4.11), $a_{n+1}^2 R_{n+1} - a_n^2 R_{n-1} = -b_n(r_{n+1} - r_n)$ and $q^{-n-\alpha}a_{n+1}^2 - q^{-n-\alpha+2}a_n^2 = r_{n+1} - qr_n + 1 - q$. We substitute (4.25), (4.61) and $b_n = \delta_n - \delta_{n+1}$, into them and obtain the following equations:

$$r_n + r_{n+1} = -q^{-n-\alpha}(\delta_n - \delta_{n+1})(q\delta_n - \delta_{n+1}) - 1 + q^{-\alpha}, \tag{4.68}$$

$$(\delta_n - \delta_{n+1})(r_{n+1} - r_n) = \frac{(q-1)q^n\delta_n(q\delta_{n-1} - \delta_n)}{q(\delta_{n-1} - \delta_{n+1})} - \frac{(q-1)q^n\delta_{n+1}(q\delta_{n+1} - \delta_{n+2})}{q(\delta_n - \delta_{n+2})}, \tag{4.69}$$

$$r_{n+1} - qr_n + 1 - q = \frac{(q-1)q^{n+1}\delta_{n+1}}{q(\delta_n - \delta_{n+2})} - \frac{(q-1)q^{n+1}\delta_n}{q(\delta_{n-1} - \delta_{n+1})}. \tag{4.70}$$

Solving the first equation for $r_{n+1}$, i.e.

$$r_{n+1} = -r_n - q^{-n-\alpha}(\delta_n - \delta_{n+1})(q\delta_n - \delta_{n+1}) - 1 + q^{-\alpha}$$

and substituting $r_{n+1}$ from it into (4.69) and (4.70), we find two equations involving $r_n$ and $\delta_{n-1}$, $\delta_n$, $\delta_{n+1}$, $\delta_{n+2}$. Eliminating $r_n$ between these equations, we get (4.66).

To prove (4.67), we use Equation (4.14).

Substituting $a_n^2$ given by (4.61), and $R_n$, $r_n$ given by (4.25), (4.60) into it, we find a new second order difference equation for $\delta_n$. □

Equation (4.66) can be used to calculate the recurrence coefficients $b_n$ recursively in terms of $b_0$ by using $\delta_n = -b_0 - b_1 - \cdots - b_{n-1}$ and the condition $\delta_{-1} = \delta_0 = 0$, $\delta_1 = -b_0$.

For instance, the recurrence coefficients $b_1$ and $b_2$ are given as follows:

$$
\begin{aligned}
b_1 &= -\frac{b_0(b_0^2+2q^{\alpha+1}-q^{\alpha+2}-1)}{b_0^2+q^{\alpha+1}-1},\\
b_2 &= \big(qb_0(q^{2+3\alpha}(1+q-q^2+q^3)-(q+1)(b_0^2-1)^2\\
&\quad + q^{\alpha}(b_0^2-1)(q^2(q+1)b_0^2-q(q^2+q+3)-1)\\
&\quad + q^{2\alpha+1}(q(3+q(2q-1))(b_0^2-1)-2)))\big)/\big((b_0^2+q^{\alpha+1}-1)\\
&\quad (q^{2\alpha+2}(q+1)-(q+1)(b_0^2-1)+2q^{\alpha+1}(q(b_0^2-1)-1))\big).
\end{aligned}
$$

### 4.2. Asymptotic expansions

System (4.66) and (4.67) can be used to study formal asymptotic behaviour of the subleading coefficient $\delta_n$ and of the recurrence coefficients via (4.61). We find the first few terms of formal expansions.

As shown in Section 3, in the classical case (for the weight $w(x) = x^{\alpha}(qx;q)_{\infty}$) we have

$$\delta_n = \frac{1}{q-1} - \frac{(1+q^{\alpha})q^n}{q-1} + \frac{q^{2n+\alpha}}{q-1}, \tag{4.71}$$

$$a_n^2 = (1-q^n-q^{n+\alpha}+q^{2n+\alpha})q^{2n+\alpha-1}, \tag{4.72}$$

$$b_n = (1+q^{\alpha})q^n-(1+q)q^{2n+\alpha}. \tag{4.73}$$

As we see, the expressions above have powers of $q^{2n}$. Let us search for solutions to (4.66) and (4.67) for $n \geq 1$ in the form of the formal series

$$\delta_n = \sum_{j=0}^{\infty} c_j q^{nj}, \quad c_0 \neq 0. \tag{4.74}$$

From Equation (4.67) we have $c_0 = \pm(q-1)^{-1}$. Note that the system (4.66), (4.67) is symmetric with respect to changing $\delta_n \to -\delta_n$. The coefficients $c_j$, $j = 0,1,2,\ldots$, can be computed explicitly in terms of $c_0$ and parameters $q$ and $\alpha$. The first few terms are as follows:

$$c_1 = \frac{-1-q^{\alpha}}{(q-1)^2c_0}, \tag{4.75}$$

$$c_2 = \frac{2q^{\alpha}}{(q-1)^2c_0}, \tag{4.76}$$

$$c_3 = -\frac{2q^{\alpha}(1+q^{\alpha})}{(q-1)^4c_0^3}, \tag{4.77}$$

$$c_4 = \frac{2q^{\alpha-1}(q+q^{2\alpha+1}+q^{\alpha}(2q+(q-1)^2(q^2+q+1)c_0^2))}{(q-1)^6c_0^5}. \tag{4.78}$$

To show similarity to the classical case of the first few terms we take

$$c_0 = \frac{1}{q-1}, \tag{4.79}$$

and get the following expansions of $\delta_n$ and of the recurrence coefficients $a_n^2$ and $b_n$:

$$\delta_n = \frac{1}{q-1} - \frac{(1+q^\alpha)q^n}{q-1} + \frac{2q^{2n+\alpha}}{q-1} - q^{3n}\left(\frac{2q^\alpha}{q-1} + \frac{2q^{2\alpha}}{q-1}\right) + \frac{2q^{\alpha-1}\left(3q^{\alpha+1} + q^{\alpha+2} + q^{2\alpha+1} + q^\alpha + q\right)q^{4n}}{q-1} + O(q^{5n}), \tag{4.80}$$

$$a_n^2 = (1 - 2q^n - 2q^{n+\alpha})q^{2n+\alpha-1} + 2q^{\alpha-2}\left(4q^{\alpha+1} + q^{\alpha+2} + q^{2\alpha+1} + q^\alpha + q\right)q^{4n} + O(q^{5n}), \tag{4.81}$$

$$b_n = (1+q^\alpha)q^n - 2q^\alpha(1+q)q^{2n} - 2q^\alpha(1+q+q^2)(1+q^\alpha)q^{3n} - 2q^{\alpha-1}\left(q^3+q^2+q+1\right)\left(3q^{\alpha+1} + q^{\alpha+2} + q^{2\alpha+1} + q^\alpha + q\right)q^{4n} + O(q^{5n}). \tag{4.82}$$

It is interesting to note that the first terms of the expansions agree with the corresponding terms in the classical case.

### 4.3. Singularity confinement

The singularity confinement is used extensively for discrete integrable systems as a discrete analogue of the Painlevé property [17,21]. The idea is that the singularity disappears after a finite number of iterations. Let us illustrate this for system (4.62), (4.63).

Assuming that $x_{n-1} \neq 0$ and $\varepsilon$ being small, we take

$$x_n = q^{n+\frac{\alpha}{2}} + \varepsilon. \tag{4.83}$$

Note that for $x_n = q^{n+\frac{\alpha}{2}}$ the denominator in (4.63) (and in (4.62)) is zero. Iterating (4.83) and using the system we can compute $x_{n+1}$, $\delta_{n+1}$ and so on. We see that after a finite number of iterations expansions become regular in $\varepsilon$. Indeed, using the system, the following expansions in $\varepsilon$ can be obtained:

$$\begin{aligned}
x_{n+1} &= \frac{q^{n+\frac{\alpha}{2}}(q^n-1)^2(q^{n+\alpha}-1)^2}{(q^{n+\frac{\alpha}{2}}x_{n-1}-1)\varepsilon^2} + \dots,\\
x_{n+2} &= q^{2+n+\frac{\alpha}{2}} - q^2\varepsilon + \dots,\\
\delta_{n+1} &= q\delta_{n-1} - \frac{(q-1)q^{n+\frac{\alpha}{2}}\delta_n}{\varepsilon} + \dots,\\
\delta_{n+2} &= q\delta_n + \frac{(q-1)^2 q^{1+n+\frac{\alpha}{2}}(q^{n+\frac{\alpha}{2}}x_{n-1}-1)\delta_n\varepsilon}{(q^n-1)^2(q^{n+\alpha}-1)^2} + \dots.
\end{aligned}$$

Therefore, we see that the singularity is confined. If, in addition to (4.83) the numerator is (close to) zero in (4.62), we may assume that

$$\delta_n = \frac{q^{1-n-\frac{\alpha}{2}}(x_n - q^{n+\frac{\alpha}{2}})\delta_{n-1}}{(q-1)} + \varepsilon,$$

then using the system we have

$$\begin{aligned}\delta_{n+1} &= -(q-1)q^{n+\frac{\alpha}{2}},\\ \delta_{n+2} &= \left(q + \frac{q^{2-n-\frac{\alpha}{2}}\delta_{n-1}}{q-1}\right)\varepsilon + \dots.\end{aligned} \tag{4.84}$$

Therefore, we see that the singularity is also confined.

## 5. A family of generalized little $q$-Laguerre polynomials

In this part we do not consider a particular weight, but rather we deal with a family of positive weights given by

$$\frac{w(x/q)}{w(x)} = Ax^2 + Bx + C, \tag{5.1}$$

for arbitrary constants $A \neq 0,\ C \neq 1$, which are supported on the exponential lattice $\{q^n | n = 0, 1, 2, \dots\}$, $0 < q < 1$, and $w(0) = w(1/q) = 0$. Such weights were considered in [15]. The methodology used in this section is almost the same as in the previous one, so we present only main results.

By the definition of $u(x)$ (see (3.1)), such weights give rise to the function $u$ given by

$$u(x) = \frac{k_1 q}{(1-q)}\frac{1}{x} + \frac{k_2 x + k_3}{1-q}, \quad k_1,\ k_2 \neq 0, \tag{5.2}$$

with $k_1 = 1 - C,\ k_2 = -Aq,\ k_3 = -Bq$. The generalized little $q$-Laguerre weight (4.1) is a particular case with $k_1 = 1 - q^{-\alpha}$, $k_2 = q^{1-\alpha}$, $k_3 = 0$. The classical case corresponds to $k_1 = 1 - q^{-\alpha},\ k_2 = 0,\ k_3 = q^{1-\alpha}$.

In [15] it is proved that if $\tilde{A}_n(x)$ and $\tilde{B}_n(x)$ are defined by (3.14) and (3.15), then

$$\tilde{A}_n(x) = \frac{a_n R_n}{(1-q)x} + \frac{a_n k_2 q^{-n}}{1-q}, \quad \tilde{B}_n(x) = \frac{r_n}{(1-q)x}, \tag{5.3}$$

where

$$\begin{aligned}R_n &= -k_1 \textstyle\int_0^1 p_n(y)p_n(y/q)\frac{w(y)}{y}d_q y,\\ r_n &= -a_n k_1 \textstyle\int_0^1 p_n(y)p_{n-1}(y/q)\frac{w(y)}{y}d_q y.\end{aligned} \tag{5.4}$$

**Lemma 5.1 [15]:** *The following relations hold:*

$$r_{n+1} + r_n = -b_n R_n - k_1, \tag{5.5}$$

$$R_n - k_2 q^{-n} b_n - k_3 = (1-q)\sum_{j=0}^{n} R_j, \tag{5.6}$$

$$a_n^2 = \frac{q^n(r_n + 1 - q^n)}{k_2}, \tag{5.7}$$

$$a_n^2 R_n R_{n-1} = r_n(r_n + k_1), \tag{5.8}$$

$$r_n(R_{n-1} + k_2 q^{-n} b_n) = q^{n+1}R_n + q^n R_{n-1} - R_{n-1} - k_2 b_n q^{-n} - k_3. \tag{5.9}$$

**Remark 3:** Substituting $\tilde{A}_n(x), \tilde{B}_n(x)$ from (5.3) and $u(x)$ from (5.2) into Equation (3.18), we get Equations (5.7), (5.8) and

$$a_n^2 = \frac{q^n(r_n+1)(1-q)\sum_{j=0}^{n-1} R_j + k_3 r_n q^n}{(qR_n + R_{n-1})k_2}. \tag{5.10}$$

Using (5.6), we get

$$a_n^2 = \frac{q^n(r_n+1)(R_{n-1} - k_2 q^{-n+1} b_{n-1} - k_3) + k_3 r_n q^n}{(qR_n + R_{n-1})k_2}. \tag{5.11}$$

Equating this equation with (5.7),
we find the following equation:

$$r_n(R_n + k_2 q^{-n} b_{n-1}) = q^n R_n + q^{n-1} R_{n-1} - R_n - k_2 q^{-n} b_{n-1} - k_3 q^{-1}, \tag{5.12}$$

which is similar to (5.9).

It is shown in [15] that in case $k_3 = 0$ there is a similar to (4.20) equation for $x_n = (r_n+1)(1-k_1)^{-1/2}$ and in case $k_3 \neq 0$ the corresponding equation is more complicated.

In the next subsection we obtain a system of difference equations for $\delta_n$ and use this system to study some further properties.

### 5.1. Nonlinear difference equations

Similarly to the study in Section 4 we have the following statements.

**Lemma 5.2:** *The auxiliary quantity $R_n$ defined by (5.4) can be expressed in terms of $\delta_n$ as follows:*

$$R_n = k_2 q^{-n-1}(q\delta_n - \delta_{n+1}) + k_3 q^{-n-1}. \tag{5.13}$$

*Moreover,*

$$\sum_{j=0}^{n} R_j = \frac{q^{-1-n}(k_3(q^{1+n}-1) + (1-q)k_2\delta_{n+1})}{q-1}. \tag{5.14}$$

***Proof:*** Since $p_n(x) = \gamma_n P_n(x)$, from Equation (5.4) we have

$$\begin{aligned} R_n &= -k_1 \int_0^1 p_n(y) p_n(y/q) \frac{w(y)}{y} d_q y \\ &= -k_1 \gamma_n^2 \int_0^1 P_n(y) P_n(y/q) \frac{w(y)}{y} d_q y. \end{aligned} \tag{5.15}$$

Using (4.3) and substituting $u$ into the equation above, we get

$$\int_0^1 P_n(y)P_n(y/q)\frac{w(y)}{y}d_qy$$
$$= -\frac{k_2}{k_1q}\int_0^1 yP_n(y)P_n(y/q)w(y)d_qy - \frac{k_3}{k_1q}\int_0^1 P_n(y)P_n(y/q)w(y)d_qy. \tag{5.16}$$

Using (4.5) and (4.30) we have

$$\int_0^1 P_n(y)P_n(y/q)\frac{w(y)}{y}d_qy$$
$$= -\frac{k_2}{k_1q}\frac{q^{-n}}{\gamma_n^2}(q\delta_n - \delta_{n+1}) - \frac{k_3}{k_1q}\frac{1}{q^n\gamma_n^2}. \tag{5.17}$$

After a simplification, the formula for $R_n$ is proved. Another formula is easily obtained by substituting (5.13) and $b_n = \delta_n - \delta_{n+1}$ into (5.6). □

**Proposition 5.3:** *We have*

$$r_n = \frac{k_2q(q^n-1)\delta_{n-1} + k_2(q-1)q^n\delta_n + (q^n-1)(k_3 - k_2\delta_{n+1})}{k_3 + k_2q\delta_{n-1} - k_2\delta_{n+1}}, \tag{5.18}$$

$$a_n^2 = \frac{(q-1)q^{2n}\delta_n}{k_3 + k_2q\delta_{n-1} - k_2\delta_{n+1}}. \tag{5.19}$$

***Proof:*** To get $r_n(x)$ given by (5.4), we substitute (5.13) into (5.9) (or (5.12)) and use $b_n = \delta_n - \delta_{n+1}$. Another formula is obtained by substituting (5.18) into (5.7). □

**Remark 4:** The expressions for $b_n$ in terms of $r_n$ are given in case $k_3 \neq 0$ by [15, Equations (26), (27)] ($b_n^2$ is given by [15, Equation (31)] when $k_3 = 0$). They allow to express $b_n$ (or $b_n^2$) in terms of $\delta_n$ using Proposition 5.3. The formulas are cumbersome so we shall omit them.

**Proposition 5.4:** *The subleading coefficient $\delta_n$ of monic orthogonal polynomials for the weight (5.1) with $k_3 = 0$ and the solution $x_n$ of the q-discrete Painlevé V equation satisfy a system of second order difference equations given by*

$$\delta_{n+1} = \frac{q(q^n\beta - x_n)\delta_{n-1} + (q-1)q^n\beta\delta_n}{q^n\beta - x_n}, \tag{5.20}$$

$$(x_nx_{n-1} - 1)(x_nx_{n+1} - 1) = \frac{\beta^2q^{2n}(x_n-\beta)^2(x_n-\beta^{-1})^2}{(x_nq^{-n}-\beta)^2}. \tag{5.21}$$

*Here*

$$x_n = \beta(1+r_n) = \frac{1+r_n}{\sqrt{1-k_1}}, \tag{5.22}$$

*Furthermore, we also have,*

$$x_{n+1} = -x_n - q^{-n-1}\beta k_2(\delta_n - \delta_{n+1})(q\delta_n - \delta_{n+1}) + \beta^{-1} + \beta. \tag{5.23}$$

***Proof***: Using (5.18) with $k_3 = 0$, we can express $\delta_{n+1}$ in terms of $r_n$, $\delta_{n-1}$ and $\delta_n$. Then taking the change of the variables (5.22), we immediately get (5.20). Equation (5.21) is given in [15].

Substituting (5.13) and $b_n = \delta_n - \delta_{n+1}$ into (5.5), we get (5.23). □

**Theorem 5.5:** *The following equations hold for the subleading coefficient $\delta_n$ of monic orthogonal polynomials with respect to the family of weights (5.1):*

(i) *the third order difference equation, which turns out to be an algebraic equation of total degree 4 in $\delta_{n-1}$, $\delta_n$, $\delta_{n+1}$ and $\delta_{n+2}$ given by*

$$\sum_{p=0}^{1}\sum_{q=0}^{3}\sum_{r=0}^{3}\sum_{s=0}^{1} d_{p,q,r,s}\delta_{n-1}^{p}\delta_{n}^{q}\delta_{n+1}^{r}\delta_{n+2}^{s} = 0, \quad p+q+r+s \le 4, \tag{5.24}$$

*with 37 non-zero coefficients $d_{p,q,r,s}$ presented in Appendix 1;*

(ii) *the second order difference equation, which turns out to be an algebraic equation of total degree 4 in $\delta_{n-1}$, $\delta_n$ and $\delta_{n+1}$, given by*

$$\sum_{p=0}^{2}\sum_{q=0}^{3}\sum_{r=0}^{2} d_{p,q,r}\delta_{n-1}^{p}\delta_{n}^{q}\delta_{n+1}^{r} = 0, \quad p+q+r \le 4, \tag{5.25}$$

*with 23 non-zero coefficients $d_{p,q,r}$ presented in Appendix 1.*

***Proof***: Substituting (5.13), (5.18) and $b_n = \delta_n - \delta_{n+1}$ into (5.5), we get Equation (5.24). To get Equation (5.25), we use (5.8) with $a_n^2$ from (5.19), $r_n$ from (5.18) and $R_n$ from (5.13). □

As in the previous section, Equation (5.24) can be used to calculate recurrence coefficients $b_n$ in terms of $b_0$. When $k_3 = 0$, we can check that (31) in [15] holds. We present the list of the first few recurrence coefficients $b_n$ for $k_3 = 0$ in the following:

$$\begin{aligned}
b_1 &= \frac{b_0(q - k_1 q - 2q^2 + q^3 - k_2 b_0^2)}{(k_1-1)q + q^2 + k_2 b_0^2},\\
b_2 &= \big(q b_0 (q^2(q+k_1-1)^2(q^2(q-1)+k_1(q+1))\\
&\quad + k_2 q(2k_1^2(q+1) + (q-1)^2(2q^2+q+1) + k_1(q-1)(3+2q(q+1)))b_0^2\\
&\quad + k_2^2(q+1)(q^2+k_1-1)b_0^4)\big)/\big(q^2(q+1)(q+k_1-1)^3\\
&\quad + k_2 q(q+k_1-1)(2k_1(q+1)+(q-1)(3q+2))b_0^2\\
&\quad + k_2^2(2q^2+(k_1-1)q+k_1-1)b_0^4\big).
\end{aligned}$$

**Remark 5:** Let $k_1 = 1 - q^{-\alpha}$, $k_2 = q^{1-\alpha}$ and $k_3 = 0$. Then Equations (5.24) and (5.25) are consistent with (4.66) and (4.67) respectively.

### 5.2. Asymptotic expansions

As in the previous section, we can find the first few terms in the formal asymptotic behaviour of the generalized system (5.24) and (5.25). Note that the system is not symmetric with respect to $\delta_n \to -\delta_n$ unless $k_3 = 0$.

Substituting

$$\delta_n = \sum_{j=0}^{\infty} d_j q^{nj}, \quad d_0 \neq 0, \ n \geq 1, \tag{5.26}$$

into (5.24) and (5.25), we get

$$K_1 q^n + K_2 q^{2n} + K_3 q^{3n} + O(q^{4n}) = 0, \tag{5.27}$$

$$L_0 + L_1 q^n + L_2 q^{2n} + L_3 q^{3n} + O(q^{4n}) = 0, \tag{5.28}$$

where $K_j$ and $L_j$ denote the coefficients at the powers of $q^{jn}$. By setting these coefficients to zero, we can find $d_j$ successively. For $L_0 = 0$, we find three solutions for $d_0$:

$$d_0^{(1)} = \frac{k_3}{k_2(1-q)}, \tag{5.29}$$

$$d_0^{(2)} = \frac{k_3 - \sqrt{k_3^2 - 4k_2 q(k_1 - 1)}}{2k_2(1-q)}, \tag{5.30}$$

$$d_0^{(3)} = \frac{k_3 + \sqrt{k_3^2 - 4k_2 q(k_1 - 1)}}{2k_2(1-q)}. \tag{5.31}$$

However, solution $d_0^{(1)}$ later on gives a contradiction and we dismiss it. For two other choices of $d_0$ we can find consecutively other coefficients $d_j$, $j = 0, 1, 2, \ldots$, which depend on the parameters $k_1$, $k_2$, $k_3$ and $q$. For instance,

$$d_0^{(2)} = \frac{k_3 - \sqrt{k_3^2 - 4k_2 q(k_1 - 1)}}{2k_2(1-q)}, \quad d_1^{(2)} = \frac{(k_1 - 2)\left(k_3 + \sqrt{k_3^2 - 4k_2 q(k_1 - 1)}\right)}{2k_2(k_1 - 1)(q - 1)},$$

$$d_0^{(3)} = \frac{k_3 + \sqrt{k_3^2 - 4k_2 q(k_1 - 1)}}{2k_2(1-q)}, \quad d_1^{(3)} = \frac{(k_1 - 2)\left(k_3 - \sqrt{k_3^2 - 4k_2 q(k_1 - 1)}\right)}{2k_2(k_1 - 1)(q - 1)}.$$

For $k_1 = 1 - q^{-\alpha}$, $k_2 = q^{1-\alpha}$, $k_3 = 0$ we have $d_0^{(2)} = (q-1)^{-1}$ and $d_0^{(3)} = -(q-1)^{-1}$, which is consistent with previous results.

### 5.3. Singularity confinement

For system (5.20) and (5.21) we assume that $x_{n-1} \neq 0$ and take

$$x_n = q^n \beta + \varepsilon \tag{5.32}$$

with $\varepsilon$ being small. Note that for $x_n = q^n \beta$ the denominator in (5.21) (and in (5.20)) is zero. Iterating (5.32) and using the system we can compute $x_{n+1}, \delta_{n+1}$ and so on. We see that after a finite number of iterations expansions become regular in $\varepsilon$. Indeed, the following

expansions in $\varepsilon$ can be obtained from the system of equations:

$$\begin{aligned}
x_{n+1} &= \frac{q^n\beta(q^n-1)^2(q^n\beta^2-1)^2}{(q^n\beta x_{n-1}-1)\varepsilon^2}+\dots,\\
x_{n+2} &= q^{2+n}\beta - q^2\varepsilon+\dots,\\
\delta_{n+1} &= -\frac{(q-1)q^n\beta\delta_n}{\varepsilon}+q\delta_{n-1}+\dots,\\
\delta_{n+2} &= q\delta_n+\frac{(q-1)^2q^{n+1}\beta(q^n\beta x_{n-1}-1)\delta_n\varepsilon}{(q^n-1)^2(q^n\beta^2-1)^2}+\dots.
\end{aligned}$$

Therefore, we see that the singularity is confined. If, in addition to (5.32), the numerator is (close to) zero in (5.20), we have that

$$\delta_n = \frac{(q\beta - q^{1-n}x_n)\delta_{n-1}}{(1-q)\beta}+\varepsilon,$$

then using the system we have

$$\begin{aligned}
\delta_{n+1} &= -(q-1)q^n\beta,\\
\delta_{n+2} &= \left(q+\frac{q^{2-n}\delta_{n-1}}{(q-1)\beta}\right)\varepsilon+\dots.
\end{aligned} \tag{5.33}$$

Thus, we see that the singularity is also confined.

## 6. Deformation of the generalized little $q$-Laguerre weight

In this section we study the recurrence coefficients for the weight functions supported on the exponential lattice $\{q^n|n=0,1,2,\dots\}$, $0<q<1$, satisfying the $q$-difference Equation (3.1) with

$$u(x) = \frac{k_1q}{(1-q)}\frac{1}{(x+t)}+\frac{k_2x+k_3}{1-q},\quad k_1,\ k_2\neq 0,\ t>0, \tag{6.1}$$

which corresponds to the weight given by

$$\frac{w(x/q)}{w(x)} = Ax^2+Bx+1+\frac{Dx}{x+t}, \tag{6.2}$$

with $k_1=-D,\ k_2=-Aq,\ k_3=-Bq$. It is straightforward to calculate that

$$\frac{u(qx)-u(y)}{qx-y} = \frac{1}{(1-q)}\left(-\frac{k_1q}{(qx+t)(y+t)}+k_2\right). \tag{6.3}$$

Hence, the expressions (3.14) and (3.15) can be computed as follows:

$$\tilde{A}_n(x) = \frac{a_nR_nq}{(1-q)(qx+t)}+\frac{a_nk_2q^{-n}}{1-q}, \tag{6.4}$$

$$\tilde{B}_n(x) = \frac{r_nq}{(1-q)(qx+t)}, \tag{6.5}$$

where

$$R_n = -k_1 \int_0^1 p_n(y)p_n(y/q)\frac{w(y)}{y+t}d_qy, \tag{6.6}$$

$$r_n = -a_nk_1 \int_0^1 p_n(y)p_{n-1}(y/q)\frac{w(y)}{y+t}d_qy. \tag{6.7}$$

The compatibility conditions (3.16), (3.17) give rise to the following system (after comparing the coefficients at the powers of $x$):

$$r_{n+1} + r_n = b_n(-tk_2q^{-1-n} - R_n) - k_1 - tk_3q^{-1}, \tag{6.8}$$

$$R_n - k_2q^{-n}b_n - k_3 - (1-q)\sum_{j=0}^{n} R_j = 0, \tag{6.9}$$

$$a_{n+1}^2(tk_2q^{-2-n} + R_{n+1}) + a_n^2(-tk_2q^{-n} - R_{n-1}) = -b_n(r_{n+1} - r_n) - t(1-q^{-1}), \tag{6.10}$$

$$k_2a_{n+1}^2 - k_2q^2a_n^2 = q^{n+1}(r_{n+1} - qr_n + 1 - q). \tag{6.11}$$

Here we use notation $S_n = \sum_{j=0}^{n} R_j$, and (6.9) becomes

$$S_n = \frac{R_n - k_2q^{-n}b_n - k_3}{1-q}. \tag{6.12}$$

Substituting $\tilde{A}_n(x)$, $\tilde{B}_n(x)$ from (6.4), (6.5) and $u(x)$ from (6.1) into Equation (3.18), we get another system:

$$\begin{aligned} &q^n((q^nr_n(-tk_3 - k_1q) - t(-tk_2(q^n-1) - (q-1)q^nS_{n-1}) - q^{n+1}r_n^2)) \\ &+ a_n^2(-tk_2 - q^nR_{n-1})(-tk_2 - q^{n+1}R_n) = 0, \end{aligned} \tag{6.13}$$

$$\begin{aligned} &q^n((q-1)q^nS_{n-1} + r_n(-tk_2 - k_3q^n + (q-1)q^nS_{n-1}) + 2tk_2(q^n-1)) \\ &+ k_2a_n^2(2tk_2 + q^nR_{n-1} + q^{n+1}R_n) = 0, \end{aligned} \tag{6.14}$$

$$a_n^2 = \frac{q^n(r_n + 1 - q^n)}{k_2}. \tag{6.15}$$

**Lemma 6.1:** *The auxiliary quantity $R_n$ defined by (6.6) can be expressed in terms of $\delta_n$ as follows:*

$$R_n = k_2q^{-n-1}(q\delta_n - \delta_{n+1}) + k_3q^{-n-1}. \tag{6.16}$$

The proof is similar to the proof of Lemma 5.2.

**Proposition 6.2:** *The auxiliary quantity $r_n$ defined by (6.7) can be expressed in terms of $\delta_n$ as follows:*

$$r_n = \frac{k_2q(q^n-1)\delta_{n-1} + k_2(q-1)q^n\delta_n + (q^n-1)(k_3 - k_2\delta_{n+1})}{k_3 + k_2q\delta_{n-1} - k_2\delta_{n+1} - tk_2}. \tag{6.17}$$

*The recurrence coefficients $a_n^2$ is given in terms of $\delta_n$ by*

$$a_n^2 = \frac{q^n((q-1)q^n\delta_n - t(q^n-1))}{k_3 + k_2 q\delta_{n-1} - k_2\delta_{n+1} - tk_2}. \tag{6.18}$$

***Proof*:** The first equation is obtained by substituting $S_n$ from (6.12) and $a_n^2$ from (6.15) into (6.14). Then we make use of (6.16) and $b_n = \delta_n - \delta_{n+1}$. Substituting $r_n$ from (6.17) into (6.15) gives (6.18). □

**Theorem 6.3:** *The following equations hold for the subleading coefficient $\delta_n$ of monic orthogonal polynomials with respect to the function (6.1):*

(i) *the third order difference equation, which turns out to be an algebraic equation of total degree 4 in $\delta_{n-1}$, $\delta_n$, $\delta_{n+1}$ and $\delta_{n+2}$ given by*

$$\sum_{p=0}^{1}\sum_{q=0}^{3}\sum_{r=0}^{3}\sum_{s=0}^{1} h_{p,q,r,s}\delta_{n-1}^p\delta_n^q\delta_{n+1}^r\delta_{n+2}^s = 0, \quad p+q+r+s \le 4, \tag{6.19}$$

*with 37 non-zero coefficients $h_{p,q,r,s}$ presented in Appendix 1;*

(ii) *the second order difference equation, which turns out to be an algebraic equation of total degree 4 in $\delta_{n-1}$, $\delta_n$ and $\delta_{n+1}$, given by*

$$\sum_{p=0}^{2}\sum_{q=0}^{3}\sum_{r=0}^{2} h_{p,q,r}\delta_{n-1}^p\delta_n^q\delta_{n+1}^r = 0, \quad p+q+r \le 4, \tag{6.20}$$

*with 25 non-zero coefficients $h_{p,q,r}$ presented in Appendix 1.*

***Proof*:** Substituting (6.16), (6.17) and $b_n = \delta_n - \delta_{n+1}$ into (6.8), we get Equation (6.19). To get Equation (6.20), we use (6.14) with $S_n$ from (6.12) and $a_n^2$ from (6.15), then make use of (6.16), (6.17) and $b_n = \delta_n - \delta_{n+1}$. □

**Remark 6:**

(1) Let $t = 0$. Then Equations (6.19) and (6.20) are consistent with (5.24) and (5.25) respectively.

(2) We can find the first few terms in the formal asymptotic behaviour of the system (6.19) and (6.20) by substituting

$$\delta_n = \sum_{j=0}^{\infty} h_j q^{nj}, \quad h_0 \ne 0, \ n \ge 1, \tag{6.21}$$

into (6.20). Next, we get three solutions for $h_0$, which are cumbersome, but when $t = 0$, they agree with $d_0$.

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## Appendix 1.

The list of coefficients for Theorem 5.5:

$$d_{0,0,0,0} = -k_3^2 q^{1+n}(-2 + k_1 + q^n + q^{1+n}), \quad d_{0,0,0,1} = k_2 k_3 q^{1+n}(-2 + k_1 + q^n + q^{1+n}),$$
$$d_{0,0,1,0} = k_3^3 - k_2 k_3 q^{1+n}(2 - k_1 - q^n - 2q^{1+n} + q^{2+n}),$$
$$d_{0,0,1,1} = -k_2 k_3^2 - k_2^2 q^{1+n}(-2 + k_1 + q^n + q^{1+n}), \quad d_{0,0,2,0} = -2k_2 k_3^2 + k_2^2(-1 + q)q^{2+2n},$$
$$d_{0,0,2,1} = 2k_2^2 k_3, \quad d_{0,0,3,0} = k_2^2 k_3, \quad d_{0,0,3,1} = -k_2^3,$$
$$d_{0,1,0,0} = -k_3^3 - k_2 k_3 q^{1+n}((-2 + k_1)q - q^n + 2q^{1+n} + q^{2+n}),$$
$$d_{0,1,0,1} = k_2 k_3^2 + k_2^2(-1 + q)q^{1+2n}, \quad d_{0,1,1,0} = 2k_2 k_3^2(1 + q) + k_2^2 q^{2+n}(-2 + k_1 + q^n + q^{1+n}),$$
$$d_{0,1,1,1} = -k_2^2 k_3(2 + q), \quad d_{0,1,2,0} = -k_2^2(k_3 + 3k_3 q), \quad d_{0,1,2,1} = k_2^3(1 + q), \quad d_{0,1,3,0} = k_2^3 q,$$
$$d_{0,2,0,0} = -2k_2 k_3^2 q - k_2^2(-1 + q)q^{2+2n},$$
$$d_{0,2,0,1} = k_2^2 k_3 q, \quad d_{0,2,1,0} = k_2^2 k_3 q(3 + q), \quad d_{0,2,1,1} = -k_2^3 q,$$
$$d_{0,2,2,0} = -k_2^3 q(1 + q), \quad d_{0,3,0,0} = -k_2^2 k_3 q^2, \quad d_{0,3,1,0} = k_2^3 q^2,$$
$$d_{1,0,0,0} = -k_2 k_3 q^{2+n}(-2 + k_1 + q^n + q^{1+n}),$$
$$d_{1,0,0,1} = k_2^2 q^{2+n}(-2 + k_1 + q^n + q^{1+n}), \quad d_{1,0,1,0} = k_2 k_3^2 q - k_2^2(-1 + q)q^{3+2n},$$
$$d_{1,0,1,1} = -k_2^2 k_3 q, \quad d_{1,0,2,0} = -k_2^2 k_3 q, \quad d_{1,0,2,1} = k_2^3 q,$$
$$d_{1,1,0,0} = -k_2 k_3^2 q - k_2^2 q^{3+n}(-2 + k_1 + q^n + q^{1+n}),$$

$$d_{1,1,0,1} = k_2^2k_3q, \quad d_{1,1,1,0} = k_2^2k_3q(1+2q), \quad d_{1,1,1,1} = -k_2^3q(1+q), \quad d_{1,1,2,0} = -k_2^3q^2,$$
$$d_{1,2,0,0} = -2k_2^2k_3q^2, \quad d_{1,2,0,1} = k_2^3q^2, \quad d_{1,2,1,0} = k_2^3q^2(1+q), \quad d_{1,3,0,0} = -k_2^3q^3,$$

$$\begin{aligned}
&d_{0,0,0} = -k_3^2q(-1+q^n)(-1+k_1+q^n), \quad d_{0,0,1} = 2k_2k_3q(-1+q^n)(-1+k_1+q^n),\\
&d_{0,0,2} = -k_2^2q(-1+q^n)(-1+k_1+q^n),\\
&d_{0,1,0} = k_3^3(-1+q) - k_2k_3(-1+q)q^{1+n}(-2+k_1+2q^n),\\
&d_{0,1,1} = -2k_2k_3^2(-1+q) + k_2^2(-1+q)q^{1+n}(-2+k_1+2q^n),\\
&d_{0,1,2} = k_2^2k_3(-1+q), \quad d_{0,2,0} = -k_2(-1+q)^2(-k_3^2+k_2q^{1+2n}),\\
&d_{0,2,1} = -k_2^2k_3(-2+q)(-1+q), \quad d_{0,2,2} = -k_2^3(-1+q),\\
&d_{0,3,0} = -k_2^2k_3(-1+q)q, \quad d_{0,3,1} = k_2^3(-1+q)q,\\
&d_{1,0,0} = -2k_2k_3q^2(-1+q^n)(-1+k_1+q^n), \quad d_{1,0,1} = 2k_2^2q^2(-1+q^n)(-1+k_1+q^n),\\
&d_{1,1,0} = 2k_2k_3^2(-1+q)q - k_2^2(-1+q)q^{2+n}(-2+k_1+2q^n),\\
&d_{1,1,1} = -3k_2^2k_3(-1+q)q, \quad d_{1,1,2} = k_2^3(-1+q)q,\\
&d_{1,2,0} = k_2^2k_3q(1-3q+2q^2), \quad d_{1,2,1} = -k_2^3(-1+q)^2q,\\
&d_{1,3,0} = -k_2^3(-1+q)q^2, \quad d_{2,0,0} = -k_2^2q^3(-1+q^n)(-1+k_1+q^n),\\
&d_{2,1,0} = k_2^2k_3(-1+q)q^2, \quad d_{2,1,1} = -k_2^3(-1+q)q^2, \quad d_{2,2,0} = k_2^3(-1+q)q^3.
\end{aligned}$$

The list of coefficients for Theorem 6.3:

$$\begin{aligned}
h_{0,0,0,0} &= -t^3k_2^2k_3q^n - t^2k_2q^n(2k_3^2+k_1k_2q) - k_3^2q^{1+n}(-2+k_1+q^n+q^{1+n})\\
&\quad - tk_3q^n(k_3^2+k_2q(-2+2k_1+q^n+q^{1+n})),\\
h_{0,0,0,1} &= t^2k_2^2k_3q^n + k_2k_3q^{1+n}(-2+k_1+q^n+q^{1+n}) + tk_2q^n(k_3^2+k_2q(-1+k_1+q^{1+n})),\\
h_{0,0,1,0} &= t^3k_2^3 + k_3^3 + t^2k_2^2k_3(3+q^n) - k_2k_3q^{1+n}(2-k_1-q^n-2q^{1+n}+q^{2+n})\\
&\quad - tk_2(-k_3^2(3+q^n) + k_2q^{1+n}(1-k_1-q^n-q^{1+n}+q^{2+n})),\\
h_{0,0,1,1} &= -t^2k_2^3 - k_2k_3^2 - tk_2^2k_3(2+q^n) - k_2^2q^{1+n}(-2+k_1+q^n+q^{1+n}),\\
h_{0,0,2,0} &= -2t^2k_2^3 - 4tk_2^2k_3 - 2k_2k_3^2 + k_2^2(-1+q)q^{2+2n},\\
h_{0,0,2,1} &= 2tk_2^3 + 2k_2^2k_3, \quad h_{0,0,3,0} = tk_2^3 + k_2^2k_3, \quad h_{0,0,3,1} = -k_2^3,\\
h_{0,1,0,0} &= -t^3k_2^3 - k_3^3 - t^2k_2^2k_3(3+q^{1+n}) - k_2k_3q^{1+n}((-2+k_1)q - q^n + 2q^{1+n} + q^{2+n})\\
&\quad - tk_2(k_3^2(3+q^{1+n}) + k_2q^{1+n}((-1+k_1)q - q^n + q^{1+n} + q^{2+n})),\\
h_{0,1,0,1} &= t^2k_2^3 + 2tk_2^2k_3 + k_2k_3^2 + k_2^2(-1+q)q^{1+2n},\\
h_{0,1,1,0} &= 2t^2k_2^3(1+q) + 2k_2k_3^2(1+q) + tk_2^2k_3(4+4q+q^{1+n})\\
&\quad + k_2^2q^{2+n}(-2+k_1+q^n+q^{1+n}),\\
h_{0,1,1,1} &= -tk_2^3(2+q) - k_2^2k_3(2+q), \quad h_{0,1,2,0} = -tk_2^3(1+3q) - k_2^2(k_3+3k_3q),\\
h_{0,1,2,1} &= k_2^3(1+q), \quad h_{0,1,3,0} = k_2^3q,\\
h_{0,2,0,0} &= -2t^2k_2^3q - 4tk_2^2k_3q - 2k_2k_3^2q - k_2^2(-1+q)q^{2+2n},\\
h_{0,2,0,1} &= tk_2^3q + k_2^2k_3q, \quad h_{0,2,1,0} = tk_2^3q(3+q) + k_2^2k_3q(3+q),\\
h_{0,2,1,1} &= -k_2^3q, \quad h_{0,2,2,0} = -k_2^3q(1+q), \quad h_{0,3,0,0} = -tk_2^3q^2 - k_2^2k_3q^2, \quad h_{0,3,1,0} = k_2^3q^2,\\
h_{1,0,0,0} &= t^2k_2^2k_3q^{1+n} - k_2k_3q^{2+n}(-2+k_1+q^n+q^{1+n})\\
&\quad - tk_2q^{1+n}(k_3^2+k_2q(-1+k_1+q^n)),\\
h_{1,0,0,1} &= tk_2^2k_3q^{1+n} + k_2^2q^{2+n}(-2+k_1+q^n+q^{1+n}),
\end{aligned}$$

$$h_{1,0,1,0} = t^2k_2^3q + 2tk_2^2k_3q + k_2k_3^2q - k_2^2(-1+q)q^{3+2n},$$
$$h_{1,0,1,1} = -tk_2^3q - k_2^2k_3q, \quad h_{1,0,2,0} = -tk_2^3q - k_2^2k_3q, \quad h_{1,0,2,1} = k_2^3q,$$
$$h_{1,1,0,0} = -t^2k_2^3q - k_2k_3^2q - tk_2^2k_3q(2+q^{1+n}) - k_2^2q^{3+n}(-2+k_1+q^n+q^{1+n}),$$
$$h_{1,1,0,1} = tk_2^3q + k_2^2k_3q, \quad h_{1,1,1,0} = tk_2^3q(1+2q) + k_2^2k_3q(1+2q),$$
$$h_{1,1,1,1} = -k_2^3q(1+q), \quad h_{1,1,2,0} = -k_2^3q^2, \quad h_{1,2,0,0} = -2tk_2^3q^2 - 2k_2^2k_3q^2,$$
$$h_{1,2,0,1} = k_2^3q^2, \quad h_{1,2,1,0} = k_2^3q^2(1+q), \quad h_{1,3,0,0} = -k_2^3q^3,$$

$$\begin{aligned}
h_{0,0,0} &= t^2k_2^2k_3^2q^n(-1+q^n) + tk_2k_3q^n(k_3^2+k_1k_2q)(-1+q^n)\\
&\quad + k_2k_3^2q^{1+n}(-1+q^n)(-1+k_1+q^n),\\
h_{0,0,1} &= -t^2k_2^3k_3q^n(-1+q^n) - tk_2^2q^n(2k_3^2+k_1k_2q)(-1+q^n)\\
&\quad - 2k_2^2k_3q^{1+n}(-1+q^n)(-1+k_1+q^n),\\
h_{0,0,2} &= tk_2^3k_3q^n(-1+q^n) + k_2^3q^{1+n}(-1+q^n)(-1+k_1+q^n),\\
h_{0,1,0} &= t^2k_2^3k_3(-1+q)q^n(-1+q^n) + k_2^2k_3(-1+q)q^{1+2n}(-2+k_1+2q^n)\\
&\quad + tk_2^2(-1+q)q^n(k_1k_2q^{1+n} + k_3^2(-2+q^n)) - k_2k_3^3(-1+q)q^n,\\
h_{0,1,1} &= 2k_2^2k_3^2(-1+q)q^n + t^2k_2^4q(-1+q^n) - k_2^3(-1+q)q^{1+2n}(-2+k_1+2q^n)\\
&\quad - tk_2^3k_3(q+2q^n-q^{2n}-3q^{1+n}+q^{1+2n}),\\
h_{0,1,2} &= -k_2^3k_3(-1+q)q^n - tk_2^4q(-1+q^n),\\
h_{0,2,0} &= -k_2^2k_3^2(-1+q)^2q^n + k_2^3(-1+q)^2q^{1+3n} - t^2k_2^4(-q+q^n-q^{1+n}+q^{2+n})\\
&\quad - tk_2^3k_3(-q+2q^n-3q^{1+n}+2q^{2+n}),\\
h_{0,2,1} &= k_2^3k_3q^n(2-3q+q^2) + tk_2^4(-q+2q^n-2q^{1+n}+q^{2+n}), \quad h_{0,2,2} = k_2^4(-1+q)q^n,\\
h_{0,3,0} &= tk_2^4(-1+q)q^{1+n} + k_2^3k_3(-1+q)q^{1+n}, \quad h_{0,3,1} = -k_2^4(-1+q)q^{1+n},\\
h_{1,0,0} &= t^2k_2^3k_3q^{1+n}(-1+q^n) + tk_2^2q^{1+n}(2k_3^2+k_1k_2q)(-1+q^n)\\
&\quad + 2k_2^2k_3q^{2+n}(-1+q^n)(-1+k_1+q^n),\\
h_{1,0,1} &= -t^2k_2^4q(-1+q^n) - 2k_2^3q^{2+n}(-1+q^n)(-1+k_1+q^n) - tk_2^3k_3q(-1-q^n+2q^{2n})),\\
h_{1,0,2} &= tk_2^4q(-1+q^n),\\
h_{1,1,0} &= -2k_2^2k_3^2(-1+q)q^{1+n} + t^2k_2^4q(-1+q^n) + k_2^3(-1+q)q^{2+2n}(-2+k_1+2q^n)\\
&\quad + tk_2^3k_3q(-1+3q^n-q^{2n}-2q^{1+n}+q^{1+2n}),\\
h_{1,1,1} &= 3k_2^3k_3(-1+q)q^{1+n} + tk_2^4(-1+q)q(-1+2q^n), \quad h_{1,1,2} = -k_2^4(-1+q)q^{1+n},\\
h_{1,2,0} &= -k_2^3k_3q^{1+n}(1-3q+2q^2) - tk_2^4q(-q+q^n-2q^{1+n}+2q^{2+n}),\\
h_{1,2,1} &= k_2^4(-1+q)^2q^{1+n}, \quad h_{1,3,0} = k_2^4(-1+q)q^{2+n},\\
h_{2,0,0} &= tk_2^3k_3q^{2+n}(-1+q^n) + k_2^3q^{3+n}(-1+q^n)(-1+k_1+q^n),\\
h_{2,0,1} &= -tk_2^4q^2(-1+q^n), \quad h_{2,1,0} = -k_2^3k_3(-1+q)q^{2+n} + tk_2^4q^2(-1+q^n),\\
h_{2,1,1} &= k_2^4(-1+q)q^{2+n}, \quad h_{2,2,0} = -k_2^4(-1+q)q^{3+n}.
\end{aligned}$$